



# A new topological approach to the $L^\infty$ -uniqueness of operators and the $L^1$ -uniqueness of Fokker–Planck equations<sup>☆</sup>

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## Abstract

The usual semigroups of kernels on a Polish space  $E$  are in general not strongly continuous on  $L^\infty(E, \mu)$  with respect to the norm topology. We introduce a new topology on  $L^\infty(E, \mu)$  such that they become  $C_0$ -semigroups for which we can establish a simplified Hille–Yosida theorem. The new topology will allow us to introduce the uniqueness of pre-generator on  $L^\infty(E, \mu)$  which turns out to be equivalent to the  $L^1$ -uniqueness of the associated Fokker–Planck equation among many others, and it is intimately related with the Liouville properties for  $L^1$ -harmonic functions. The uniqueness of several second order elliptic differential operators in  $L^\infty$  are studied: (1) one-dimensional diffusion operators  $a(x)f'' + b(x)f'$ ; (2) Schrödinger operators  $-(1/2)\Delta + V$ ; (3) multi-dimensional diffusion generator  $(1/2)\Delta + \beta \cdot \nabla$ .

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**Keywords:** Uniqueness of operators in  $L^\infty$ ; Fokker–Planck equations; Schrödinger operators; Diffusions; Liouville property

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## 0. Introduction

Let us begin with an example which motivates the studies in this paper. Consider the Schrödinger operator  $-(1/2)\Delta + V$ , defined on  $\mathcal{D} = C_0^\infty(\mathbb{R}^d)$  ( $d \geq 1$ ), where the potential  $V \in L_{\text{loc}}^\infty(\mathbb{R}^d)$  such that  $V^-$  belongs to the Kato class. The uniqueness of  $-(1/2)\Delta + V$  in  $L^2(\mathbb{R}^d, dx)$ , defined as the essential self-adjointness, is equivalent to the uniqueness of  $L^2$ -solution of the Schrödinger equation or of the associated heat equation  $\partial_t u = ((1/2)\Delta - V)u$  with  $u(0) = f \in L^2$  fixed, in the distribution sense. Under the previous assumption on  $V$ , it is known that  $(-(1/2)\Delta + V, \mathcal{D})$  is essentially self-adjoint (see [9, Corollary 2.2, p. 17]), and the unique  $L^2$  solution of the heat equation is given by the famous Feynman–Kac semigroup:

$$u(t, x) = P_t^V f(x) := \mathbb{E}^x f(B_t) \exp\left(-\int_0^t V(B_s) ds\right) \quad (0.1)$$

where  $(B_t)$  is the Brownian motion in  $\mathbb{R}^d$  defined on some probability space  $(\Omega, \mathcal{F}, (\mathbb{P}_x)_{x \in \mathbb{R}^d})$  with  $\mathbb{P}_x(B_0 = x) = 1, \forall x \in \mathbb{R}^d$ , and  $\mathbb{E}^x$  means the expectation with respect to (in short: w.r.t.)  $\mathbb{P}_x$ .

Remark that  $(P_t^V)$  is a semigroup of bounded operators on  $L^p(\mathbb{R}^d, dx)$  for all  $1 \leq p \leq \infty$ , which is strongly continuous for  $1 \leq p < \infty$ , but never strongly continuous in  $L^\infty$ . Indeed a general result due to Lotz says that the generator of a strongly continuous semigroup on  $L^\infty$  w.r.t. its norm  $\|\cdot\|_\infty$  is always bounded.

How to define the uniqueness of  $(-(1/2)\Delta + V, \mathcal{D})$  in  $L^p$ ? In the classical situation where  $p \in [1, +\infty)$ , the  $L^p$ -uniqueness of  $(-(1/2)\Delta + V, \mathcal{D})$  for  $1 \leq p < +\infty$  is defined as the following [2,14,42,43]: the closure of  $(\Delta/2 - V, \mathcal{D})$  in  $L^p$  coincides with the generator  $\mathcal{L}_{(p)}$  of  $(P_t^V)$ .

This definition is no longer valid for  $p = +\infty$  because of the non-strong continuity of  $(P_t^V)$  in  $L^\infty$ .

Hence to obtain a correct definition of  $L^\infty$ -uniqueness, we should change the topology of  $L^\infty$  so that  $(P_t^V)$  is strongly continuous. The good topology on  $L^\infty$  seems to be the topology  $\mathcal{C}(L^\infty, L^1)$  of uniform convergence over the compact subsets of  $L^1$ . Indeed we can prove the Phillips type theorem:

*If and only if  $(P_t)$  is a  $C_0$ -semigroup on  $L^1$ , its dual  $(P_t^*)$  is a strongly continuous semigroup of continuous operators on  $L^\infty$  w.r.t.  $\mathcal{C}(L^\infty, L^1)$ .*

That is why we are led to the following

**Definition 0.1.** A linear operator  $A$  on  $L^\infty(E, \mu)$  with domain  $\mathcal{D}$  is essentially a generator, if  $A$  is closable w.r.t.  $\mathcal{C}(L^\infty, L^1)$  and its closure w.r.t.  $\mathcal{C}(L^\infty, L^1)$  is the generator of some  $C_0$ -semigroup  $(T_t)$  on  $(L^\infty, \mathcal{C}(L^\infty, L^1))$ . In that case we say that  $A$  or  $-A$  is  $L^\infty$ -unique.

This notion is natural and useful, illustrated by

**Theorem 0.2.** Let  $A$  be a linear operator on  $L^\infty(E, \mu)$  with domain  $\mathcal{D}$ . Assume that there is a  $C_0$ -semigroup  $(T_t)$  on  $(L^\infty, \mathcal{C}(L^\infty, L^1))$  such that its generator  $\mathcal{L}$  is an extension of  $A$  (the existence assumption). Then the following properties are all equivalent:

- (a)  $A$  is  $L^\infty$ -unique;
- (b) the closure of  $A$  w.r.t.  $\mathcal{C}(L^\infty, L^1)$  is exactly  $\mathcal{L}$ ;
- (c)  $A^*$  is the generator of some  $C_0$ -semigroup on  $L^1$ ;
- (d)  $A^* = \mathcal{L}^*$ , the generator of the dual  $C_0$ -semigroup  $(T_t^*)$  on  $L^1$ ;
- (e) for some  $\lambda > \lambda_0 := \lim_{t \rightarrow \infty} \frac{1}{t} \log \|T_t\|_\infty$ , the range  $(\lambda - A)(\mathcal{D})$  is dense in  $L^\infty$  w.r.t.  $\mathcal{C}(L^\infty, L^1)$ ;
- (f) (Liouville property) for some (or for all)  $\lambda > \lambda_0$ ,

$$\text{if } f \in L^1 \text{ satisfies } (\lambda - A^*)f = 0 \quad \Rightarrow \quad f = 0; \quad (0.2)$$

- (g) ( $L^1$ -uniqueness for the Fokker–Planck equation) for every  $f \in L^1$ , the Fokker–Planck equation

$$\partial_t u(t) = A^* u(t); \quad u(0) = f \quad (0.3)$$

has a unique solution in the following sense:

- (g.i)  $t \rightarrow u(t)$  is continuous from  $\mathbb{R}^+$  to  $L^1$ ,
- (g.ii) for every  $h \in \mathcal{D}$ ,  $\langle u(t) - f, h \rangle = \int_0^t \langle u(s), Ah \rangle ds$ .

And the unique solution is given by  $u(t) := T_t^* f$ .

- (h)  $(P_t)$  is the unique  $C_0$ -semigroup on  $(L^\infty, \mathcal{C}(L^\infty, L^1))$  such that its generator extends  $A$ .

When  $A$  is a second order elliptic type differential operator defined on  $\mathcal{D} = C_0^\infty(\mathbb{R}^d)$ , the solutions of the resolvent equation (0.2) and the Fokker–Planck equation (0.3) are exactly weak solutions in the distribution sense. And the proofs of the uniqueness of solutions of those two equations are often separated and different in the current works. The theorem above shows that they are in fact equivalent.

The  $L^1$ -uniqueness for the Fokker–Planck equation (0.3) is much more delicate than the  $L^p$ -uniqueness for  $1 < p < +\infty$ . For example, for the Laplacian  $\Delta$  on a complete connected Riemannian manifold  $M$ , S.T. Yau [47,48] and Strichartz [38] proved that  $\Delta$  is always  $L^p$ -unique for  $1 < p < +\infty$ , Davies [11] proved that  $\Delta$  is  $L^1$ -unique if and only if (iff)  $M$  is stochastically complete (i.e., the Brownian motion on  $M$  does not explode); but for the  $L^\infty$ -uniqueness of  $\Delta$  or equivalently the  $L^1$ -uniqueness of the associated Fokker–Planck equation, Azencott [3] and Li, Schoen [26] constructed several counter-examples for which  $\text{Ric}(x) \sim -cd(x, x_0)^{2+\varepsilon}$  or  $M$  is stochastically complete, but the  $L^1$ -uniqueness of the Fokker–Planck equation (associated with  $\Delta$ ) fails. The sharpest sufficient condition in this context was found by P. Li [25]: the  $L^1$ -uniqueness of the Fokker–Planck equation (associated with  $\Delta$ ) holds true once if  $\text{Ric}(x) \geq -c(1 + d(x, x_0)^2)$  for some constant  $c > 0$ .

Let us now explain the meaning of the  $L^1$ -uniqueness of the Fokker–Planck equation in the case where  $A$  is a second order elliptic differential operator defined on  $\mathcal{D} = C_0^\infty(\mathbb{R}^d)$ . That  $L^1$ -uniqueness might seem at the first glance purely as a game of mathematicians. This is not true, indeed the  $L^1$ -uniqueness of the Fokker–Planck equation has important physical meaning: if the initial distribution of the heat is  $f(x)$ , then the solution  $u(t, x)$  to the Fokker–Planck equation is the distribution of the heat at time  $t$  and at the position  $x$ , so  $\int |u(t, x)| dx$  (the  $L^1$ -norm) is the total heat or energy in the system at time  $t$  (usually  $\int u(t, x)^2 dx$  or  $\int |\nabla_x u(t, x)|^2 dx$  are called “energy” in parallel to other physical models, but those are not the physical energy in the Fokker–Planck equation). That is why the  $L^1$ -uniqueness of the Fokker–Planck equation or the  $L^\infty$ -uniqueness of  $A$  is really of physical importance.

This paper is organized as follows. In the next section we obtain a new version of the Philips theorem for the dual semigroup of a  $C_0$ -semigroup on a general locally convex vector space. That is the basis of our study on  $L^\infty$ . In Section 2 we introduce the uniqueness of a linear operator  $A$  on a general locally convex vector space, unifying the  $L^p$ -uniqueness and  $L^\infty$ -uniqueness. A series of equivalent characterizations extending Theorem 0.2 are furnished. Section 3 is devoted to the study of the  $C_0$ -semigroups on  $L^\infty$  w.r.t.  $\mathcal{C}(L^\infty, L^1)$ : different easily checkable characterizations are provided and a simplified Hille–Yosida theorem is established for the sub-Markov semigroups. Furthermore, several examples based on known results are presented for illustrating the difference of  $L^\infty$ -uniqueness of an operator from its  $L^p$ -uniqueness.

The remained part of this paper consists to illustrate how those general results can be applied for several important operators. In Section 4 we study the one-dimensional diffusion operator  $A = a(x)\frac{d^2}{dx^2} + b(x)\frac{d}{dx}$  defined on  $\mathcal{D} = C_0^\infty(x_0, y_0)$ : it is shown that the  $L^\infty$ -uniqueness of  $A$  is equivalent to the fact that  $x_0, y_0$  are *no entrance boundary* in the classification of Feller (i.e., the heat at the boundary  $\{x_0, y_0\}$  cannot enter into  $(x_0, y_0)$ ), and it is also equivalent to the Liouville property for the integrable nonnegative  $A$ -subharmonic functions. This condition should be compared with the characterization of the  $L^1$ -uniqueness of  $A$  in [43]:  $A$  is  $L^1$ -unique (or equivalently the  $L^\infty$ -uniqueness for the Fokker–Planck equation holds true), iff  $\{x_0, y_0\}$  are no accessible (i.e., the heat inside  $(x_0, y_0)$  cannot hit the boundary  $\{x_0, y_0\}$ ). The characterization of the  $L^\infty$ -uniqueness of  $A$  here is a natural continuation of the works by Wielens [41], Eberle [14] and Djellout [13] on the  $L^p$ -uniqueness ( $1 < p < +\infty$ ).

Section 5 is devoted to the proof of the following result. *If the potential  $V$  belongs locally to  $L^\infty(\mathbb{R}^d, dx)$  and  $V^-$  belongs to the Kato class, then  $(-(1/2)\Delta + V, C_0^\infty(\mathbb{R}^d))$  is  $L^\infty$ -unique.* This is just an extension of Kato's theorem from  $L^2$  to  $L^\infty$  and our proof is based on the mean value property of subharmonic functions established by Aizenman, Simon [1].

In the last section (Section 6) we exploit the idea of “no entrance boundary” for multi-dimensional diffusion operator  $A = \frac{1}{2}\Delta + b(x) \cdot \nabla$  defined on  $\mathcal{D} = C_0^\infty(\mathbb{R}^d)$ . This is realized by comparison with an one-dimensional diffusion operator. Several sharp results, better than the known ones about the  $L^1$ -Liouville property are also provided.

Main results of this paper were announced in [46].

## 1. A new variant of Phillips theorem

The Hille–Yosida theory of semigroups of operators is generalized from the Banach space to a general locally convex (l.c. in short) vector space, by L. Schwartz [35] and Yosida [49]. The purpose of this section is to introduce a new topology  $\mathcal{C}(X^*, X)$  on the dual space  $X^*$  of a l.c. vector space  $X$  such that the dual semigroup  $(T_t^*)$  of a  $C_0$ -semigroup  $(T_t)$  on  $X$  becomes a  $C_0$ -semigroup on  $X^*$  with respect to (w.r.t. in short) that topology (a variant of Phillips theorem about dual semigroups); and to establish a Hille–Yosida theorem for  $C_0$ -semigroups (w.r.t.  $\mathcal{C}(X^*, X)$ ) of contractions on the dual  $X^*$  of some Banach space  $X$ . This paper illustrates once more the usefulness and the necessity of the theory of  $C_0$ -semigroups on a general l.c. vector space.

### 1.1. $C_0$ -semigroup: definition and several lemmas

Let  $X$  be a real linear vector space endowed with some locally convex Hausdorff topology  $\beta$ . To emphasize the role of  $\beta$ , we write often  $X_\beta$  instead of  $X$ . Following [49, p. 234], we introduce

**Definition 1.1.** Given a family of continuous linear operators  $(T_t)_{t \geq 0}$  on  $X_\beta$ . It is called a  $C_0$ -semigroup on  $X$ , if:

- (i)  $T_0 = I$ ;  $T_s T_t = T_{s+t}$  for all  $s, t \geq 0$  (semigroup property);
- (ii)  $\forall x \in X, t \rightarrow T_t x$  is continuous from  $\mathbb{R}^+$  to  $X_\beta$  (strong continuity);
- (iii) for some  $\lambda_0 \in \mathbb{R}$ ,  $(e^{-\lambda_0 t} T_t)_{t \geq 0}$  are equicontinuous (notice that,  $X_\beta$  being not necessarily metrizable, the equicontinuity here should be interpreted by means of semi-norm as in [49, p. 234]).

The generator  $\mathcal{L}$  of  $(T_t)$  with domain  $\mathbb{D}(\mathcal{L})$  is defined as

$$\mathbb{D}(\mathcal{L}) = \left\{ x \in X; \lim_{t \rightarrow 0} \frac{1}{t} (T_t x - x) = z \text{ exists in } X_\beta \right\},$$

$$\mathcal{L}x := \lim_{t \rightarrow 0} \frac{1}{t} (T_t x - x), \quad \forall x \in \mathbb{D}(\mathcal{L}). \quad (1.1)$$

Throughout this paper we assume that  $X_\beta$  is sequentially complete. In that case  $\mathcal{L}$  is a densely defined, closed operator, and its resolvent

$$(\lambda - \mathcal{L})^{-1}x = \int_0^\infty e^{-\lambda t} T_t x dt, \quad \forall \lambda > \lambda_0, x \in X,$$

is well defined and continuous on  $X$  (see Yosida [49, Chapter IX]).

Let  $Y := X_\beta^*$ , the topological dual space of  $X_\beta$ . The dual bilinear relation for  $(x, y) \in X \times Y$  will be denoted by  $\langle x, y \rangle$ . Throughout this paper, the dual (or adjoint) operator of a linear operator  $A$  on  $X$  (respectively on  $Y$ ) with domain  $\mathbb{D}(A)$  will be always taken w.r.t. the weak topology  $\sigma(X, Y)$  (respectively  $\sigma(Y, X)$ ). For example, if  $B$  is a linear operator on  $Y$  with domain  $\mathbb{D}(B)$  dense in  $(Y, \sigma(Y, X))$  (but not-necessarily dense in  $Y$  w.r.t. the strong dual topology), its dual operator is defined by

$$x \in \mathbb{D}(B^*) \quad \text{iff} \quad \exists \tilde{x} \in X: \langle x, By \rangle = \langle \tilde{x}, y \rangle, \quad \forall y \in \mathbb{D}(B) \text{ and } B^*x := \tilde{x}.$$

With this convention, we have

**Lemma 1.1.** [34, Chapter IV, Theorem 7.1, p. 155] *Let  $A$  be a linear operator with domain  $\mathbb{D}(A)$  dense in  $(X, \sigma(X, Y))$ . Then  $\mathbb{D}(A^*)$  is dense in  $(Y, \sigma(Y, X))$  iff  $A$  is closable w.r.t.  $\sigma(X, Y)$ . In that case,  $A^{**} = \bar{A}$ , the closure of  $A$  w.r.t.  $\sigma(X, Y)$  (which coincides with its closure in  $X_\beta$  by Hahn–Banach theorem).*

**Lemma 1.2.** [49, Chapter IX, §13, Proposition 2] *Let  $(T_t)$  be a  $C_0$ -semigroup on  $X_\beta$ . Then for any  $\lambda > \lambda_0$  ( $\lambda_0$  being determined by Definition 1.1(iii) for  $(T_t)$ ),*

$$(\lambda - \mathcal{L}^*)^{-1} = ((\lambda - \mathcal{L})^{-1})^*.$$

The following result says that the usual (strong) generator coincides with the weak generator (it is well known in the Banach space setting).

**Lemma 1.3.** *Let  $(T_t)$  be a  $C_0$ -semigroup on  $X_\beta$  which is sequentially complete. Then*

$$\mathbb{D}(\mathcal{L}) = \left\{ x \in X; \frac{1}{t}(T_t x - x) \text{ converges, as } t \rightarrow 0+, \text{ in } (X, \sigma(X, Y)) \right\}.$$

**Proof.** Let us denote the set at the right-hand side of the equality above by  $\mathbb{D}(\mathcal{L}^w)$ , and by  $\mathcal{L}^w x$  the weak  $\sigma(X, Y)$ -limit of  $(T_\varepsilon x - x)/\varepsilon$  as  $\varepsilon \rightarrow 0+$ , for each  $x \in \mathbb{D}(\mathcal{L}^w)$ . Fix  $x \in \mathbb{D}(\mathcal{L}^w)$ . For any  $t \geq 0$ , by the continuity of  $T_t$  on  $(X, \sigma(X, Y))$  (indeed, any continuous linear operator on  $X_\beta$  is continuous w.r.t. the weak topology  $\sigma(X, Y)$ ; its proof is left to the reader),

$$\lim_{\varepsilon \rightarrow 0+} \frac{1}{\varepsilon}(T_\varepsilon T_t x - T_t x) = T_t \left[ \lim_{\varepsilon \rightarrow 0+} (T_\varepsilon x - x)/\varepsilon \right] = T_t \mathcal{L}^w x,$$

where it follows  $T_t x \in \mathbb{D}(\mathcal{L}^w)$  and  $\mathcal{L}^w T_t x = T_t \mathcal{L}^w x$ . Thus for any  $y \in Y$ ,  $t \rightarrow \langle T_t x, y \rangle$  is continuous and right-derivable and its right derivative, being equal to  $\langle T_t \mathcal{L}^w x, y \rangle$ , is continuous on  $\mathbb{R}^+$ .

Thus by a well-known lemma in analysis [49, Chapter IX, p. 239] the Newton–Leibniz formula below holds:

$$\langle T_t x - x, y \rangle = \int_0^t \langle T_s \mathcal{L}^w x, y \rangle ds = \left\langle \int_0^t T_s \mathcal{L}^w x ds, y \right\rangle.$$

As  $y \in Y$  is arbitrary,  $T_t x - x = \int_0^t T_s \mathcal{L}^w x ds$  for any  $t \geq 0$ . That implies, by the continuity of  $t \rightarrow T_t(\mathcal{L}^w x)$  on  $X_\beta$ , that  $x \in \mathbb{D}(\mathcal{L})$  and  $\mathcal{L}x = \mathcal{L}^w x$ .  $\square$

## 1.2. A new version of Phillips theorem

In general, for a  $C_0$ -semigroup  $(T_t)$  on  $X$ , its dual semigroup  $(T_t^*)$  is no longer strongly continuous on the topological dual space  $Y = X^*$  w.r.t. the strong dual topology of  $Y$ . But by the famous Phillips theorem (see [49, Chapter IX, p. 273]), if  $X$  and  $Y$  are sequentially complete,  $(T_t^*)$  is a  $C_0$ -semigroup on the closure  $Y^+$  of the domain  $\mathbb{D}(\mathcal{L}^*)$  of the dual operator  $\mathcal{L}^*$  w.r.t. the strong dual topology. Moreover, the generator of  $(T_t^*)$  restricted to  $Y^+$  coincide with the restriction of  $\mathcal{L}^*$  to the set  $\{y \in \mathbb{D}(\mathcal{L}^*); \mathcal{L}^* y \in Y^+\}$ .

If  $X$  is reflexive (for example,  $X = L^p$  with  $p \in (1, +\infty)$ ), then  $Y^+ = Y$  and the Phillips theorem becomes very practical. Otherwise, characterizing  $\mathbb{D}(\mathcal{L}^*)$  and its closure  $Y^+$  is a very difficult and delicate question for concrete examples: for instance, how to determine the closure of the domain  $\mathbb{D}(\Delta)$  of the Laplacian  $\Delta$  in  $L^\infty(M, dx)$  (as the dual of the generator of the Brownian semigroup on  $L^1$ ) on a complete connected Riemannian manifold?

Our purpose is to find a locally convex topology on the dual space  $Y = X^*$  such that  $(T_t^*)$  becomes a  $C_0$ -semigroup with generator  $\mathcal{L}^*$ , to circumvent the last very difficult problem. Besides the strong dual topology (i.e., the topology of uniform convergence over  $\sigma(X, Y)$ -bounded subsets of  $X$ ), the two most often used topologies on  $Y$  are:

- the *weak topology*  $\sigma(Y, X)$ . It is also the weakest l.c. topology with respect to which the dual of  $Y$  is  $X$ ;
- the *Mackey topology*  $\tau(Y, X)$ . It is the strongest l.c. topology with respect to which the dual of  $Y$  is  $X$ . By Mackey's theorem (see [34, Chapter IV, Section 3, Corollary 1]), it is the topology of uniform convergence over the  $\sigma(X, Y)$ -compact, convex, circled subsets of  $X$ .

After much effort, we find that they are not very well adapted for our purpose above. Our effort leads to

- the *topology of uniform convergence on compact subsets of  $X_\beta$* , denoted by  $\mathcal{C}(Y, X)$ . More precisely, for an arbitrary point  $y_0 \in Y$ , a basis of neighborhoods of  $y_0$  w.r.t.  $\mathcal{C}(Y, X)$  is given by

$$N(y_0; \mathbf{K}, \varepsilon) := \left\{ y \in Y; \sup_{x \in \mathbf{K}} |\langle x, y \rangle - \langle x, y_0 \rangle| < \varepsilon \right\}, \quad (1.2)$$

where  $\mathbf{K}$  runs over all compact subsets of  $X_\beta$  and  $\varepsilon > 0$ .

It meets exactly our hope, as claimed by the following result which is a satisfactory variant of the Phillips theorem.

**Theorem 1.4.** *If  $(T_t)$  is a  $C_0$ -semigroup on  $X_\beta$ , then its dual semigroup  $(T_t^*)$  is a  $C_0$ -semigroup on  $Y = X_\beta^*$  w.r.t.  $\mathcal{C}(Y, X)$ . Moreover, the dual operator  $\mathcal{L}^*$  of  $\mathcal{L}$  is the generator of  $(T_t^*)$  on  $(Y, \mathcal{C}(Y, X))$ , and  $\mathbb{D}(\mathcal{L}^*)$  is dense in  $Y$  w.r.t. the Mackey topology  $\tau(Y, X)$ , and*

$$\begin{aligned}\mathbb{D}(\mathcal{L}^*) &= \left\{ y \in Y; \lim_{t \rightarrow 0} \frac{1}{t} (T_t^* y - y) \text{ exists w.r.t. } \mathcal{C}(Y, X) \right\} \\ &= \left\{ y \in Y; \lim_{t \rightarrow 0} \frac{1}{t} (T_t^* y - y) \text{ exists w.r.t. } \sigma(Y, X) \right\}.\end{aligned}\quad (1.3)$$

*If  $X_\beta$  is, moreover, quasi-complete (i.e., any bounded and closed subset of  $X_\beta$  is complete for Cauchy filters (instead of sequences)), then  $\mathbb{D}(\mathcal{L}^*)$  is dense in  $(Y, \mathcal{C}(Y, X))$ .*

We begin with

**Lemma 1.5.** *If  $A$  is a continuous linear mapping on  $X_\beta$ , then its dual operator  $A^*$  is continuous on  $Y = X^*$  simultaneously for the strong dual topology, the weak topology  $\sigma(Y, X)$ , the Mackey topology  $\tau(Y, X)$ , and the topology  $\mathcal{C}(Y, X)$  (of uniform convergence over compact subsets of  $X_\beta$ ).*

**Proof.** The continuity of  $A^*$  w.r.t. the strong dual topology is well known. For its continuity w.r.t.  $\sigma(Y, X)$ , see Schafer [34, Chapter IV, Theorem 7.4, p. 158]. As the dual of  $(Y, \sigma(Y, X))$  is  $X$ , the continuity of  $A^*$  w.r.t.  $\sigma(Y, X)$  implies its continuity w.r.t. the Mackey topology  $\tau(Y, X)$ , again by [34, Chapter IV, Theorem 7.4].

For the continuity of  $A^*$  w.r.t.  $\mathcal{C}(Y, X)$ , let  $\mathbf{K}$  be any compact subset in  $X$ , we have

$$\sup_{x \in \mathbf{K}} |\langle x, A^* y \rangle| = \sup_{x \in \mathbf{K}} |\langle Ax, y \rangle| = \sup_{x \in A(\mathbf{K})} |\langle x, y \rangle|.$$

Since  $A(\mathbf{K})$  is compact, then  $A^*$  is continuous w.r.t.  $\mathcal{C}(Y, X)$ .  $\square$

**Lemma 1.6.** *If  $X_\beta$  is quasi-complete (i.e., the bounded and closed subsets of  $X_\beta$  are complete), then  $\mathcal{C}(Y, X)$  is weaker than the Mackey topology  $\tau(Y, X)$ . In particular,  $(Y, \mathcal{C}(Y, X))^* = X$ .*

**Proof.** Let  $B$  be a compact subset of  $X_\beta$ , and  $C$  its closed, convex, circled hull of  $B$ . Thus  $C$  is bounded, then complete w.r.t. the strong topology  $\beta$  (by the quasi-completeness of  $X_\beta$ ), thus complete w.r.t. the (stronger) Mackey topology  $\tau(X, Y)$ . By [34, Chapter IV, Theorem 11.5, p. 189],  $C$  is compact in  $X_\beta$ . Consequently  $\mathcal{C}(Y, X)$  is weaker than the topology of uniform convergence over the  $\sigma(X, Y)$ -compact, convex, circled subsets of  $X$ , i.e., the Mackey topology  $\tau(Y, X)$ .

Since  $\mathcal{C}(Y, X)$  is also stronger than  $\sigma(Y, X)$ , then the dual of  $(Y, \mathcal{C}(Y, X))$  coincides with  $X$ .  $\square$

**Proof of Theorem 1.4.** Without loss of generality we can assume that  $(T_t)$  is equicontinuous and  $T_t x \rightarrow 0$  as  $t \rightarrow +\infty$  (otherwise consider  $(e^{-\lambda t} T_t)$  instead of  $(T_t)$  for some  $\lambda > \lambda_0$ ). By Lemma 1.5,  $(T_t^*)$  is a semigroup of continuous linear operators on  $(Y, \mathcal{C}(Y, X))$ . For clarity, we divide the remained proof into six points.



(1) *Strong continuity of  $(T_t^*)$ .* Fix  $y \in Y$  and  $s \geq 0$ . For any compact subset  $\mathbf{K}$  in  $X_\beta$  and  $t \geq 0$ ,

$$\sup_{x \in \mathbf{K}} |\langle x, (T_t^* - T_s^*)y \rangle| = \sup_{x \in \mathbf{K}} |\langle (T_t - T_s)x, y \rangle|.$$

As  $t \rightarrow s$ ,  $(T_t - T_s)x \rightarrow 0$  in  $X_\beta$  for any  $x$  by the strong continuity of  $(T_t)$ . But since  $(T_t)$  is equicontinuous, this pointwise convergence is uniform for  $x$  belonging to the compact  $\mathbf{K}$ . Thus the equality above yields

$$\lim_{t \rightarrow s} \sup_{x \in \mathbf{K}} |\langle x, (T_t^* - T_s^*)y \rangle| = 0,$$

the desired strong continuity of  $(T_t^*)$  w.r.t.  $\mathcal{C}(Y, X)$ .

(2) *Equicontinuity of  $(T_t^*)$ .* It consists to prove that for any neighborhood  $N(0; \mathbf{K}, \varepsilon)$  given by (1.2), there is a neighborhood  $N(0; \mathbf{K}', \delta)$  of the same type, such that

$$y \in N(0; \mathbf{K}', \delta) \Rightarrow T_t^* y \in N(0; \mathbf{K}, \varepsilon) \quad \text{for all } t \geq 0. \quad (1.4)$$

To this end notice that for every  $t \geq 0$ ,

$$\sup_{x \in \mathbf{K}} |\langle x, T_t^* y \rangle| = \sup_{x \in \mathbf{K}} |\langle T_t x, y \rangle| \leq \sup_{x \in \bigcup_{t \geq 0} T_t(\mathbf{K})} |\langle x, y \rangle|.$$

Consider the mapping

$$g(t, x) := T_t x, \quad g(\infty, x) = 0$$

from  $[0, +\infty] \times X$  to  $X$  ( $+\infty$  is the one point compactification of  $\mathbb{R}^+$ ). It is continuous in  $t$  and equicontinuous in  $x \in X$ , then jointly in  $(t, x) \in [0, \infty] \times X$ . Thus

$$\mathbf{K}' := g([0, +\infty] \times \mathbf{K})$$

is compact and contains  $\bigcup_{t \geq 0} T_t(\mathbf{K})$ . Now  $N(0, \mathbf{K}', \varepsilon)$  satisfies (1.4).

We have so proved  $(T_t^*)$  is a  $C_0$ -semigroup on  $(Y, \sigma(Y, X))$ . In the points below, we assume the sequential completeness of  $X_\beta$ .

(3) We show in this point that  $\mathcal{L}^* \subset \mathcal{L}'$ , where  $\mathcal{L}'$  is the generator of  $(T_t^*)$  in  $(Y, \mathcal{C}(Y, X))$  and  $\mathcal{L}^*$  is the dual operator of  $\mathcal{L}$ . (Note: by the sequential completeness,  $\mathbb{D}(\mathcal{L})$  is dense in  $X_\beta$  [49], and then its dual  $\mathcal{L}^*$  is well defined.)

In fact let  $y \in \mathbb{D}(\mathcal{L}^*)$ . For any  $x \in \mathbb{D}(\mathcal{L})$ ,

$$\langle x, T_t^* y - y \rangle = \langle T_t x - x, y \rangle = \left\langle \int_0^t \mathcal{L} T_s x \, ds, y \right\rangle = \left\langle \int_0^t T_s x \, ds, \mathcal{L}^* y \right\rangle. \quad (1.5)$$

Because  $\mathbb{D}(\mathcal{L})$  is dense in  $X$ , and the first and last terms in (1.5) are continuous in  $x \in X_\beta$ , they coincide for all  $x \in X$ . Consequently for any compact  $\mathbf{K}$  of  $X$ ,

$$\lim_{t \rightarrow 0} \sup_{x \in \mathbf{K}} \left| \left\langle x, \frac{1}{t} (T_t^* y - y) - \mathcal{L}^* y \right\rangle \right| = \lim_{t \rightarrow 0} \sup_{x \in \mathbf{K}} \left| \left\langle x - \frac{1}{t} \int_0^t T_s x \, ds, \mathcal{L}^* y \right\rangle \right|.$$

This last limit is zero, for  $T_t x \rightarrow x$  uniformly over the compact  $\mathbf{K}$ . Therefore  $y \in \mathbb{D}(\mathcal{L}')$  and  $\mathcal{L}' y = \mathcal{L}^* y$ . That proves  $\mathcal{L}^* \subset \mathcal{L}'$ .

(4) We say that  $y \in \mathbb{D}(\mathcal{L}'')$  iff  $\lim_{t \rightarrow 0} \frac{1}{t}(T_t^* y - y)$  exists w.r.t.  $\sigma(Y, X)$ , and define  $\mathcal{L}'' y$  as this weak limit for  $y \in \mathbb{D}(\mathcal{L}'')$ .

We show now the inverse inclusion  $\mathcal{L}'' \subset \mathcal{L}^*$ . Let  $y \in \mathbb{D}(\mathcal{L}'')$ . For any  $x \in \mathbb{D}(\mathcal{L})$ ,

$$\langle x, \mathcal{L}'' y \rangle = \lim_{t \rightarrow 0} \frac{1}{t} \langle x, T_t^* y - y \rangle = \lim_{t \rightarrow 0} \frac{1}{t} \langle T_t x - x, y \rangle = \langle \mathcal{L} x, y \rangle.$$

Thus  $y \in \mathbb{D}(\mathcal{L}^*)$  and  $\mathcal{L}^* y = \mathcal{L}'' y$ .

Combining that fact with the point (3), we have  $\mathcal{L}^* \subset \mathcal{L}' \subset \mathcal{L}'' \subset \mathcal{L}^*$ . Thus (1.3) is true.

(5) *Denseness of  $\mathbb{D}(\mathcal{L}^*)$  in  $(Y, \tau(Y, X))$ .* Since  $\mathcal{L}$  is a densely defined closed operator, the domain  $\mathbb{D}(\mathcal{L}^*)$  is dense in  $(Y, \sigma(Y, X))$  by Lemma 1.1. But  $\mathbb{D}(\mathcal{L}^*)$ , being a vector subspace, is then dense in  $Y$  w.r.t.  $\tau(Y, X)$  by Hahn–Banach theorem (see [34, Chapter IV, Theorem 3.1, p. 130]).

(6) *Denseness of  $\mathbb{D}(\mathcal{L}^*)$  in  $(Y, \mathcal{C}(Y, X))$  under the quasi-completeness assumption of  $X_\beta$ .* Since  $\mathbb{D}(\mathcal{L}^*)$  is dense in  $(Y, \tau(Y, X))$  as noted in point (5), then in  $(Y, \mathcal{C}(Y, X))$ , for  $\mathcal{C}(Y, X)$  is weaker than  $\tau(Y, X)$  by Lemma 1.6.  $\square$

The above result takes a very pleasant form in the following special setting (which will be very useful later).

**Corollary 1.7.** *Let  $(X, \|\cdot\|)$  be a Banach space. Assume that  $(T_t)_{t \geq 0}$  is a family of bounded operators on  $(X, \|\cdot\|)$ . Then the following properties are equivalent:*

- (a)  $(T_t)$  is a  $C_0$ -semigroup on  $(X, \|\cdot\|)$ ;
- (b) its dual semigroup  $(T_t^*)$  is a  $C_0$ -semigroup on  $(X^*, \mathcal{C}(X^*, X))$ ;
- (c)  $(T_t^*)$  is a semigroup of continuous operators on  $(X^*, \mathcal{C}(X^*, X))$  such that

$$\lim_{t \rightarrow 0+} T_t^* y = y \quad \text{w.r.t. the weak* topology } \sigma(X^*, X);$$

- (d)  $(T_t^*)$  is a semigroup of bounded operators on  $(X^*, \|\cdot\|^*)$  such that

$$\lim_{t \rightarrow 0+} T_t^* y = y \quad \text{w.r.t. } \sigma(X^*, X), \quad \forall y \in X^*.$$

In that case, the dual  $\mathcal{L}^*$  of the generator  $\mathcal{L}$  of  $(T_t)$  is exactly the generator of  $(T_t^*)$  on  $(X^*, \mathcal{C}(X^*, X))$  and

$$\begin{aligned} \mathbb{D}(\mathcal{L}^*) &= \left\{ y \in X^*; \lim_{t \rightarrow 0} \frac{1}{t} (T_t^* y - y) \text{ exists in } (X^*, \mathcal{C}(X^*, X)) \right\} \\ &= \left\{ y \in X^*; \lim_{t \rightarrow 0} \frac{1}{t} (T_t^* y - y) \text{ exists in } (X^*, \sigma(X^*, X)) \right\} \\ &= \left\{ y \in X^*; \liminf_{t \rightarrow 0} \frac{1}{t} \|T_t^* y - y\|^* < +\infty \right\}. \end{aligned} \tag{1.6}$$

Moreover,  $(e^{-\lambda t} T_t^*)$  is equicontinuous on  $(X^*, \mathcal{C}(X^*, X))$  for any  $\lambda > \lambda_0$ , where

$$\lambda_0 := \lim_{t \rightarrow \infty} \frac{1}{t} \log \|T_t\| = \lim_{t \rightarrow \infty} \frac{1}{t} \log \|T_t^*\|^*. \quad (1.7)$$

**Proof.** Below we write  $Y = X^*$  for simplicity.

(a)  $\Rightarrow$  (b): It is contained in Theorem 1.4.

(b)  $\Rightarrow$  (c): Trivial.

(c)  $\Rightarrow$  (d): For each  $t \geq 0$ , the graph  $G(T_t^*)$ , being closed in  $Y \times Y$  w.r.t.  $\mathcal{C}(Y, X) \times \mathcal{C}(Y, X)$ , is closed on the product Banach space  $(Y, \|\cdot\|) \times (Y, \|\cdot\|^*)$  (for the strong topology  $\|\cdot\|^*$  is stronger than  $\mathcal{C}(Y, X)$ ). By the closed graph theorem,  $T_t^*$  is continuous on  $(Y, \|\cdot\|^*)$ .

(d)  $\Rightarrow$  (a): Obviously  $(T_t) = (T_t^{**})$  is a semigroup of bounded operators on  $(X, \|\cdot\|)$ . In further for any  $x \in X$ ,  $\lim_{t \rightarrow 0} \langle T_t x, y \rangle = \lim_{t \rightarrow 0} \langle x, T_t^* y \rangle = \langle x, y \rangle$  for all  $y \in Y = X^*$ . By [49, Theorem, p. 233],  $(T_t)$  is then strongly continuous on  $(X, \|\cdot\|)$ .

By Theorem 1.4,  $\mathcal{L}^*$  is the generator of  $(T_t^*)$  on  $(Y, \mathcal{C}(Y, X))$ . By (1.3), the first two equalities in (1.6) are true. The third one is proved in Davies [10, Lemma 1.38, p. 26].

Finally the equality (1.7) is obvious (as  $\|T_t\| = \|T_t^*\|^*$ ), and the equicontinuity of  $(e^{-\lambda t} T_t^*)$  on  $(X^*, \mathcal{C}(X^*, X))$  follows from that of  $(e^{-\lambda t} T_t)$  on  $(X, \|\cdot\|)$  by point (2) in the proof of Theorem 1.4.  $\square$

**Remark 1.8.** This corollary shows that even in the Banach space setting, other topologies are required to study dual semigroups from the point of view of  $C_0$ -semigroups (and the powerful Hille–Yosida theorem).

**Remark 1.9.** To get feeling about the difference between the quasi-completeness and sequential completeness, we quote an example:  $l^1$  endowed with the weak topology  $\sigma(l^1, l^\infty)$  is sequentially complete, but not quasi-complete.

### 1.3. Some properties of the topology $\mathcal{C}(X^*, X)$

To better understand the topology  $\mathcal{C}(X^*, X)$ , we notice the following facts:

**Lemma 1.10.** Let  $(X, \|\cdot\|)$  be a Banach space, and denote the topology  $\mathcal{C}(X^*, X)$  by  $\mathcal{C}$ . We have

- (a) the dual space  $(X^*, \mathcal{C}(X^*, X))^*$  of  $(X^*, \mathcal{C}(X^*, X))$  is  $X$ .
- (b) Any bounded subset of  $(X^*, \mathcal{C}(X^*, X))$  is  $\|\cdot\|^*$ -bounded. And restricted to a  $\|\cdot\|^*$ -bounded subset of  $X^*, \mathcal{C}(X^*, X)$  coincides with  $\sigma(X^*, X)$ .
- (c)  $(X^*, \mathcal{C}(X^*, X))$  is complete.
- (d)  $\mathcal{C}(X, X_C^*)$  coincides with the  $\|\cdot\|$ -topology on  $X$ .

**Proof.** (a) Since  $(X, \|\cdot\|)$  is complete, this part is a particular case of Lemma 1.6.

(b) If  $\mathbf{B}$  is a bounded subset of  $(X^*, \mathcal{C}(X^*, X))$ , then for any  $x \in X$ , the set of real numbers  $\{\langle x, y \rangle; y \in \mathbf{B}\}$  is bounded. By the uniform boundedness theorem of Banach,  $\sup_{y \in \mathbf{B}} \sup_{x \in X: \|x\| \leq 1} \|\langle x, y \rangle\| = \sup_{y \in \mathbf{B}} \|y\|^* < +\infty$ , where the first claim follows.

Let  $\mathbf{B}$  be a  $\|\cdot\|^*$ -bounded subset of  $X^*$ , and  $(y_\alpha)_{\alpha \in I} \subset \mathbf{B}$  a directed family converging to  $y \in \mathbf{B}$  in the weak\* convergence topology  $\sigma(X^*, X)$ . Then the direct family of linear functionals  $\{x \rightarrow \langle x, y_\alpha \rangle; \alpha \in I\}$  converges pointwise to  $x \rightarrow \langle x, y \rangle$  (for all  $x \in X$ ), and it is equicontinuous

on  $(X, \|\cdot\|)$ , hence converges uniformly over any compact subsets of  $(X, \|\cdot\|)$  (by Ascoli–Arzelà theorem). Thus  $y_\alpha \rightarrow y$  in the topology  $\mathcal{C}(X^*, X)$ , i.e., the second claim in (b) is true.

(c) As  $(X, \|\cdot\|)$  is a Banach space, the family of all compact subsets of  $X$  is *saturated* in the sense of Schafer [34, Chapter III, p. 81]. By Grothendieck theorem for the characterization of the completeness of the topology of uniform convergence over a saturated family [34, Chapter IV, Theorem 6.2], it is enough to verify that if a linear functional  $f: X \rightarrow \mathbb{R}$  is continuous on every compact subset  $K$  of  $X$ , then  $f$  is continuous on  $X$  (i.e.,  $f \in X^*$ ). Indeed let  $(x_n)$  be a sequence converging to zero in  $X$ . The set  $K := \{x_n\} \cup \{0\}$  is compact. By the continuity of  $f$  on  $K$ , we have  $\lim_{n \rightarrow \infty} f(x_n) = 0$ , the desired continuity of  $f$  on  $X$ .

(d) Recall that the  $\|\cdot\|$ -topology on  $X$  is the topology of uniform convergence over the  $\|\cdot\|^*$ -bounded subsets of  $X^*$ . Since the compact subsets of  $(X^*, \mathcal{C}(X^*, X))$ , being compact w.r.t.  $\sigma(X^*, X)$ , are  $\|\cdot\|^*$ -bounded, then the  $\|\cdot\|$ -topology on  $X$  is stronger than the topology  $\mathcal{C}(X, X_C^*)$  of uniform convergence over the  $\mathcal{C}(X^*, X)$ -compact subsets of  $X^*$ .

Conversely on the unit closed ball  $\mathbf{B}$  of  $(X^*, \|\cdot\|^*)$ , the topologies  $\mathcal{C}(X^*, X)$  and  $\sigma(X^*, X)$  coincide on  $\mathbf{B}$ . As  $\mathbf{B}$  is  $\sigma(X^*, X)$ -compact,  $\mathbf{B}$  is  $\mathcal{C}(X^*, X)$ -compact. Hence the  $\|\cdot\|$ -topology, being the topology of uniform convergence over  $\mathbf{B}$ , is weaker than  $\mathcal{C}(X, X_C^*)$ .  $\square$

**Remark 1.11.** In the above setting, the closed ball  $\mathbf{B}^*(0, r)$  of  $X^*$  is a  $\mathcal{C}(X^*, X)$ -closed, convex, circled subset absorbing any bounded subset of  $X^*$ . But  $\mathbf{B}^*(0, r)$  is not a  $\mathcal{C}(X^*, X)$ -neighborhood of 0. Hence  $(X^*, \mathcal{C}(X^*, X))$  is neither bornological (then non-metrizable), nor barrelled (then non-Baire). Thus some fundamental results in functional analysis, such as the closed graph theorem, are not available on  $(X^*, \mathcal{C}(X^*, X))$ : we should be very careful with this topology.

**Lemma 1.12.** Let  $(X, \|\cdot\|)$  be a Banach space. The following properties are equivalent for a linear operator  $A: X^* \rightarrow X^*$  with domain  $\mathbb{D}(A) = X^*$ :

- (i)  $A$  is continuous w.r.t.  $\mathcal{C}(X^*, X)$ ;
- (i')  $A$  is continuous w.r.t.  $\sigma(X^*, X)$ ;
- (i'')  $A$  is continuous w.r.t.  $\tau(X^*, X)$ ;
- (ii)  $A$  is  $\|\cdot\|^*$ -bounded and  $A^{s*}(X) \subset X$ , where  $A^{s*}: X^{**} \rightarrow X^{**}$  is the strong adjoint operator on the bi-dual space  $X^{**}(\supset X)$ ;
- (iii)  $A^{s*}(X) \subset X$ ;
- (iv) there is some bounded operator  $B$  on  $(X, \|\cdot\|)$  such that  $A = B^*$ .

**Proof.** (i)  $\Rightarrow$  (ii). If  $A$  is continuous w.r.t.  $\mathcal{C}(X^*, X)$ , then the images set of a  $\mathcal{C}(X^*, X)$ -bounded subset by  $A$  is  $\mathcal{C}(X^*, X)$ -bounded. By part (b) of Lemma 1.10,  $A$  is then  $\|\cdot\|^*$ -bounded. Moreover, since the dual of  $(X^*, \mathcal{C}(X^*, X))$  is  $X$  (by part (a) of Lemma 1.10), the adjoint  $A^*$  is an everywhere defined continuous operator on  $(X, \tau(X, X^{**}))$  by Lemma 1.5. That implies  $A^{s*}x = A^*x \in X$  for any  $x \in X$ .

(i') or (i'')  $\Rightarrow$  (ii). The same proof as in (i)  $\Rightarrow$  (ii).

(ii)  $\Rightarrow$  (iii). Trivial.

(iii)  $\Rightarrow$  (iv). By (iii),  $B := A^{s*}|_X$  is an everywhere defined linear operator on  $X$ , and  $B = A^*$  by definition of  $A^*$ .  $B$  is then closed on  $(X, \|\cdot\|)$ , thus continuous on  $(X, \|\cdot\|)$  by the closed graph theorem. By Lemma 1.1,  $A^{**} = B^* \supset A$ . But  $A$  is everywhere defined on  $X^*$ , then  $B^* = A$ .

(iv)  $\Rightarrow$  (i), (i'), (i''). It is a direct consequence of Lemma 1.5.  $\square$

**Remark 1.13.** For verifying a semigroup  $(T_t)$  to be a  $C_0$ -semigroup, the equicontinuity of  $(e^{-\lambda_0 t} T_t)$  (= property (iii) in Definition 2.1) is often the most difficult. But notice that it is an automatic consequence of properties (i) and (ii) in Definition 2.1 in the following situations:

- (1)  $(X, \beta) = (X, \|\cdot\|)$  is a Banach space (well known);
- (2)  $(X, \beta) = (Y^*, \mathcal{C}(Y^*, Y))$ , where  $Y$  is a Banach space (by (c)  $\Rightarrow$  (b) in Corollary 1.7 and Lemma 1.12 (i)  $\Rightarrow$  (iv)).

**Example 1.14.** Let  $\Delta$  be the Laplacian–Beltrami operator in the distribution sense on a complete Riemannian manifold  $M$ . Let  $(T_t)$  be the semigroup of the Brownian motion  $(B_t)_{t \geq 0}$  (with explosion time  $e$ ) on  $M$  with generator  $\Delta$  (it corresponds to the minimal fundamental solution in PDE's theory).  $(T_t)$  is a  $C_0$ -semigroup on  $L^p(M, dx)$  for each  $p \in [1, +\infty)$ . But for  $p = +\infty$ ,  $(T_t)$  is not strongly continuous on  $L^\infty$  w.r.t. the norm  $\|\cdot\|_\infty$ -topology, but it is a  $C_0$ -semigroup on  $L^\infty$  w.r.t.  $\mathcal{C}(L^\infty, L^1)$  by Corollary 1.7. Below  $L^\infty(M, dx)$  is endowed with this last  $\mathcal{C}(L^\infty, L^1)$ -topology.

We denote by  $\Delta_{(p)}$  the generator of  $(T_t)$  on  $L^p(M, dx)$  for  $p \in [1, +\infty]$ . We shall see later (Section 3.4) that the equality

$$\mathbb{D}(\Delta_{(p)}) = \{f \in L^p(M, dx); \Delta f \in L^p(M, dx) \text{ (in the distribution sense)}\} \quad (1.8)$$

holds always for  $p \in (1, +\infty)$ . But for  $p = +\infty$ , that equality holds if and only if  $M$  is *stochastically complete* (i.e.,  $e = +\infty$  a.s.); and for  $p = 1$ , it holds only under some curvature condition.

#### 1.4. Hille–Yosida theorem for $C_0$ -semigroup of contractions on $(X^*, \mathcal{C}(X^*, X))$

By Hille–Yosida theorem, a linear operator  $\mathcal{L}$  on  $X_\beta$  is the generator of some  $C_0$ -semigroup  $(T_t)$  on  $X_\beta$  iff  $\mathbb{D}(\mathcal{L})$  is dense in  $X_\beta$ , the resolvent  $(1 - \lambda(\mathcal{L} - \lambda_0))^{-1}$  exists for each  $\lambda > 0$  (where  $\lambda_0$  is given in Definition 1.1), and

$$\left(1 - \frac{1}{n}(\mathcal{L} - \lambda_0)\right)^{-m}, \quad n, m \geq 1, \quad (1.9)$$

are equicontinuous on  $X_\beta$ .

Our purpose here is to find some much easier characterization (than (1.9)) of the generator of a  $C_0$ -semigroup of contractions on  $X^*$  endowed with the topology  $\mathcal{C}(X^*, X)$ . This will be crucial for the study of  $C_0$ -semigroups on  $L^\infty = (L^1)^*$  in Section 3.

**Theorem 1.15.** Let  $(X, \|\cdot\|)$  be a Banach space. A linear operator  $\mathcal{L}$  on  $X^*$  is the generator of some  $C_0$ -semigroup  $(T_t)$  on  $(X^*, \mathcal{C}(X^*, X))$  with  $\|T_t^*\| \leq 1$  (contraction), if and only if the three conditions below are satisfied:

- (i)  $\mathbb{D}(\mathcal{L})$  is dense in  $(X^*, \sigma(X^*, X))$  and  $\mathcal{L}$  is a closed operator on  $(X^*, \mathcal{C}(X^*, X))$ ;
- (ii) if  $\lambda > 0$ ,  $y \in \mathbb{D}(\mathcal{L})$ ,  $y - \lambda \mathcal{L}y = z$ , then

$$\|y\|^* \leq \|z\|^*;$$

- (iii) the range of  $1 - \lambda \mathcal{L}$  is  $X^*$  for any  $\lambda > 0$ .

**Proof.** The necessity is obvious. We establish the sufficiency.

By (i), Lemma 1.6 and Hahn–Banach theorem,  $\mathbb{D}(\mathcal{L})$  is dense in  $(X^*, \mathcal{C}(X^*, X))$ . By (ii),  $1 - \lambda\mathcal{L} : \mathbb{D}(\mathcal{L}) \rightarrow X^*$  is injective. But it is also surjective by condition (iii). Then  $(1 - \lambda\mathcal{L})^{-1}$  exists on  $X^*$  and it is  $\|\cdot\|^*$ -bounded (by (ii)).

On the other hand,  $\mathcal{L}$  is also a densely defined closed operator on  $(X^*, \sigma(X^*, X))$  by Lemma 1.10 and Hahn–Banach theorem. Therefore  $\mathcal{L}^*$  is a densely defined closed operator on  $(X, \sigma(X, X^*))$ , by Lemma 1.1. Again by Hahn–Banach theorem,  $\mathcal{L}^*$  is a densely defined closed operator on  $(X, \|\cdot\|)$ . By Phillips theorem for the resolvent of a dual operator on Banach space [49, Chapter VIII, Section 6],  $(1 - \lambda\mathcal{L}^*)^{-1}$  is bounded on  $X$  iff  $(1 - \lambda\mathcal{L}^{**})^{-1}$  is bounded on  $X^*$  and in that case,

$$[(1 - \lambda\mathcal{L}^*)^{-1}]^* = (1 - \lambda\mathcal{L}^{**})^{-1}.$$

But  $\mathcal{L}^{**} = \mathcal{L}$  by Lemma 1.1, the condition of the “if” part in the Phillips theorem above is satisfied by what we have shown above. Consequently we obtain for all  $\lambda > 0$ ,

$$\|(1 - \lambda\mathcal{L}^*)^{-1}\| = \|(1 - \lambda\mathcal{L})^{-1}\|^* \leq 1.$$

Applying the Hille–Yosida theorem on  $X$ ,  $\mathcal{L}^*$  is the generator of some  $C_0$ -semigroup  $(S_t)$  of contractions on  $X$ . By Theorem 1.4 or Corollary 1.7,  $\mathcal{L} = (\mathcal{L}^*)^*$  is the generator of the  $C_0$ -semigroup  $(T_t = S_t^*)$  on  $(X^*, \mathcal{C}(X^*, X))$  such that  $\|T_t\|_\infty \leq 1$ .  $\square$

## 2. Uniqueness of operators in a locally convex vector space

In this section we introduce the uniqueness of a linear operator  $A$  on a general locally convex vector space. A series of equivalent characterizations of it are furnished, unifying in some sense many known (and independent) results in the literature in the Banach space case.

### 2.1. General case

Let  $X_\beta = (X, \beta)$  be a locally convex sequentially complete vector space. As suggested by Arendt [2, 1986], Eberle [14, 1997] and the first author [42, Lemma 2.6], we introduce

**Definition 2.1.** Let  $A : X \rightarrow X$  be a linear operator with domain  $\mathcal{D}$  dense in  $X_\beta$ .  $A$  is said to be a *pre-generator*, if there exists some  $C_0$ -semigroup on  $X_\beta$  such that its generator  $\mathcal{L}$  extends  $A$ .

$A$  is said to be *essentially a generator* in  $X_\beta$ , if  $A$  is closable and its closure  $\bar{A}$  w.r.t.  $\beta$  is the generator of some  $C_0$ -semigroup on  $X_\beta$ .

In that case we say also that  $A$  or  $-A$  is  $X_\beta$ -unique.

The following result justifies that the above definition of uniqueness is not abusive and in fact very natural.

**Theorem 2.1.** Let  $Y = X^*$  (the topological dual space of  $X_\beta$ ) and  $A$  a linear operator on  $X$  with domain  $\mathcal{D}$  (it is often the test-functions space), which is assumed to be dense in  $X_\beta$ . Assume that there is a  $C_0$ -semigroup  $(T_t)$  on  $X_\beta$  such that its generator  $\mathcal{L}$  is an extension of  $A$  (the existence assumption). Let  $\lambda_0$  be the constant in Definition 1.1(iii) for  $(T_t)$ . Then the following properties are all equivalent:

- (i)  $A$  is essentially a generator in  $X_\beta$  (or  $X_\beta$ -unique).
- (ii) The closure of  $A$  in  $X_\beta$  is exactly  $\mathcal{L}$ .
- (iii)  $A^* = \mathcal{L}^*$  which is the generator of the dual  $C_0$ -semigroup  $(T_t^*)$  on  $(Y, \mathcal{C}(Y, X))$ .
- (iv) For some  $\lambda > \lambda_0$ , the range  $(\lambda - A)(\mathcal{D})$  is dense in  $X_\beta$ .
- (v) (Liouville property) For some (or equivalently for all)  $\lambda > \lambda_0$ ,  $\text{Ker}(\lambda - A^*) = \{0\}$ , i.e.,

$$\text{if } y \in \mathbb{D}(A^*) \text{ satisfies } (\lambda - A^*)y = 0 \quad \Rightarrow \quad y = 0. \quad (2.1)$$

- (v') For all  $\lambda > \lambda_0$  and for all  $y \in Y$ , the resolvent equation (2.2) has a unique solution  $z \in \mathbb{D}(A^*)$ :

$$(\lambda - A^*)z = y \quad (2.2)$$

and the unique solution is given by  $z = ((\lambda - \mathcal{L})^{-1})^* y = (\lambda - \mathcal{L}^*)^{-1} y$ .

- (vi) (Uniqueness of strong solutions for the Cauchy problem) For each  $x \in \mathbb{D}(\bar{A})$ , there is a unique strong solution  $v(t)$  of

$$\partial_t v(t) = \bar{A}v(t), \quad v(0) = x. \quad (2.3)$$

(I.e.,  $t \mapsto v(t)$  is differentiable from  $\mathbb{R}^+$  to  $X_\beta$  and its derivative  $\partial_t v(t)$  coincides with  $\bar{A}v(t)$ . And the solution is given by  $v(t) = T_t x$ .)

- (vii) (Uniqueness of weak solutions for the dual Cauchy problem) For every  $y_0 \in Y$ , the equation

$$\partial_t u(t) = A^* u(t); \quad u(0) = y_0 \quad (2.4)$$

has a unique  $\mathcal{C}(Y, X_\beta)$ -continuous weak solution ( $u(t) \in Y$ ); more precisely there is a unique  $(t \mapsto u(t) \in Y)$  satisfying:

- (vii.1)  $t \mapsto u(t)$  is continuous from  $\mathbb{R}^+$  to  $(Y, \mathcal{C}(Y, X_\beta))$  (but we do not assume  $u(t) \in \mathbb{D}(A^*)$ );

(vii.2) for every  $x \in \mathcal{D}$ ,  $\langle x, u(t) - y_0 \rangle = \int_0^t \langle Ax, u(s) \rangle ds$ .

And the unique solution is given by  $u(t) := T_t^* y_0$ .

- (vii')  $A^*$  is the generator of some  $C_0$ -semigroup on  $(Y, \mathcal{C}(Y, X))$ .

- (viii) There is only one  $C_0$ -semigroup such that its generator extends  $A$ .

**Remark 2.2.** This result is partially known in the Banach space setting, see Arendt [2], Pazy [32], Davies [11] and the first named author [42,43]. Notice that only uniqueness of strong solution of the Cauchy problem (2.3) is developed systematically in the theory of  $C_0$ -semigroups, but in this work we are much more interested by the dual Cauchy problem (2.4), which becomes exactly the Fokker–Planck equation describing the evolution of heat distribution when  $A$  is a second order elliptic differential operator. The weak solutions defined in this theorem for the resolvent equation (2.2) and for the dual Cauchy problem (2.4) correspond exactly to those in the distribution sense in the usual theory of partial differential equations.

An important fact showing the subtleness of the uniqueness problem is: without the existence assumption made in this theorem, even in the Banach space setting, the existence and the uniqueness of strong solution of (2.3) is not sufficient to the  $X_\beta$ -uniqueness of  $A$  (one should impose furthermore the continuous dependence on the initial condition, see Arendt [2]).

**Remark 2.3.** When  $(S(t))$  is a  $C_0$ -semigroup on  $(Y, \mathcal{C}(Y, X))$  such that its generator  $B$  is contained in  $A^*$  (i.e.,  $B \subset A^*$  or the restriction of  $A^*$  to the domain  $\mathbb{D}(B)$  coincides with  $B$ ), according to Feller [16, p. 473], we call that  $\mathbb{D}(B)$  is a boundary (or lateral) condition for  $A^*$  or for the dual equation (2.4).

This terminology can be justified as follows: if  $u(t)$  is a  $\mathcal{C}(Y, X)$ -continuous weak solution of (2.4) such that the “boundary” condition  $u(t) \in \mathbb{D}(B)$  ( $\forall t \geq 0$ ) is satisfied, then  $u(t) = S(t)u(0)$ . This claim follows by (v) (applied to  $A := B$ ).

By Theorem 1.4, property (viii) is equivalent to

(viii') *There is only one  $C_0$ -semigroup on  $(Y, \mathcal{C}(Y, X))$  such that its generator is contained in  $A^*$ .*

In the terminology of Feller above, (viii') is equivalent to

(viii'') *There is only one boundary condition  $\mathbb{D}(B)$  for  $A^*$  or (2.4).*

A standard example is the Laplacian  $\Delta$  on a bounded and open domain  $D \subset \mathbb{R}^d$  with smooth boundary  $\partial D$ : let  $A = \Delta$  with  $\mathbb{D}(A) = C_0^\infty(D)$  (the space of real infinitely differentiable functions on  $D$  with compact support), acting on  $X = L^2(D, dx) = X^* = Y$ . Then

$$\begin{aligned}\mathbb{D}(A^*) &= \{f \in L^2(D, dx); \Delta f \in L^2(D, dx) \text{ in distribution}\} = H^2(D), \\ A^* f &= \Delta f, \quad \forall f \in \mathbb{D}(A^*).\end{aligned}$$

There are many boundary conditions: (1)  $\Delta_{\text{Dirichlet}} = A^*|_{H^2(D) \cap H_0^1(D)}$  (the Dirichlet boundary); (2)  $\mathbb{D}(\Delta_{\text{Neumann}}) = \{f \in H^2; \partial_{\mathbf{n}} f|_{\partial D} = 0 \text{ (in distribution)}\}$  where  $\mathbf{n}$  is the outer normal to  $\partial D$ , etc.

We begin with the following lemma which is well known for  $k = 1$  in the Banach space setting (see, e.g., [2, Corollary 1.32] or [10, Theorem 1.9], but their proofs involve essentially the Banach space structure).

**Lemma 2.4.** *Let  $(T_t)$  be a  $C_0$ -semigroup on  $X_\beta$  with generator  $\mathcal{L}$ . Given an integer  $k \geq 1$  assume that  $\mathcal{D}$  is a linear subspace of  $\mathbb{D}(\mathcal{L}^k)$ , dense in  $X_\beta$ . If  $\mathcal{D}$  is stable by  $(T_t)_{t \geq 0}$ , then  $\mathcal{D}$  is a core for  $\mathcal{L}^k$  (i.e., the closure of  $\mathcal{L}^k|_{\mathcal{D}}$  is  $\mathcal{L}^k$ ).*

**Proof.** We present a very elementary proof by following [45, Lemma 3.5]. Fix some  $\lambda > \lambda_0$ , where  $(e^{-\lambda_0 t} T_t)$  is equicontinuous. Let  $A = \mathcal{L}|_{\mathcal{D}}$ , the restriction of  $\mathcal{L}$  to  $\mathcal{D}$ . Let us prove that  $(\lambda - A)^k(\mathcal{D})$  is dense in  $X_\beta$ .

For this purpose, by Hahn–Banach theorem, it is enough to establish that if  $y_0 \in Y$  satisfies

$$\langle (\lambda - A)^k x, y_0 \rangle = 0, \quad \forall x \in \mathcal{D},$$

then  $y_0 = 0$ . Indeed fix such  $y_0$ . We have for any  $x \in \mathcal{D} \subset \mathbb{D}(\mathcal{L}^k)$  and  $t \geq 0$ ,

$$\frac{d^k}{dt^k} \langle x, e^{-\lambda t} T_t^* y_0 \rangle = \frac{d^k}{dt^k} \langle e^{-\lambda t} T_t x, y_0 \rangle = \langle (-\lambda + \mathcal{L}) e^{-\lambda t} T_t x, y_0 \rangle = 0$$



because  $e^{-\lambda t} T_t x \in \mathcal{D}$  by our assumption. Hence  $\langle x, e^{-\lambda t} T_t^* y_0 \rangle$  is a polynomial (with degree not greater than  $k-1$ ). But by our choice of  $\lambda$ ,  $e^{-\lambda t} T_t^* y_0 \rightarrow 0$  w.r.t.  $\sigma(Y, X)$  as  $t \rightarrow +\infty$ . Therefore  $\langle x, e^{-\lambda t} T_t^* y_0 \rangle = 0$  for any  $t \geq 0$  and that is for all  $x \in \mathcal{D}$ . As  $\mathcal{D}$  is dense in  $X_\beta$ , we get  $y_0 = 0$ , the desired result.

Now for any  $x \in \mathbb{D}(\mathcal{L}^k)$ , by the claim just shown, there is a directed family  $(x_\alpha)$  in  $\mathcal{D}$  such that

$$(\lambda - A)^k x_\alpha \rightarrow (\lambda - \mathcal{L})^k x.$$

By the continuity of  $(\lambda - \mathcal{L})^{-k}$  we have

$$x_\alpha = (\lambda - \mathcal{L})^{-k} [(\lambda - A)^k x_\alpha] \rightarrow (\lambda - \mathcal{L})^{-k} (\lambda - \mathcal{L})^k x = x,$$

and by the continuity of  $\mathcal{L}^k (\lambda - \mathcal{L})^{-k} = (\lambda (\lambda - \mathcal{L})^{-1} - I)^k$  we also have

$$\mathcal{L}^k x_\alpha = \mathcal{L}^k (\lambda - \mathcal{L})^{-k} [(\lambda - A)^k x_\alpha] \rightarrow \mathcal{L}^k (\lambda - \mathcal{L})^{-k} [(\lambda - \mathcal{L})^k x] = \mathcal{L}^k x.$$

Whence the closure of  $A^k$  is  $\mathcal{L}^k$ , the desired claim.  $\square$

### Proof of Theorem 2.1.

(1) The cycle (i)  $\Leftrightarrow$  (ii)  $\Leftrightarrow$  (iii)  $\Leftrightarrow$  (iv).

(i)  $\Rightarrow$  (iv). By (i),  $\bar{A}$  is the generator of some  $C_0$ -semigroup  $(T'_t)$  on  $X_\beta$ . Let  $\lambda' \in \mathbb{R}$  so that  $(e^{-\lambda' t} T'_t)$  is equicontinuous. It is well known that for any  $\lambda > \lambda'$ , the resolvent  $(\lambda - \bar{A})^{-1}$  is continuous on  $X_\beta$  and in particular  $(\lambda - \bar{A})(\mathbb{D}(\bar{A})) = X$ . This implies that  $(\lambda - A)(\mathcal{D})$  is dense in  $X_\beta$ .

(iv)  $\Rightarrow$  (iii). By (iv) and the Hahn–Banach theorem,  $\text{Ker}(\lambda - A^*) = \{0\}$ . By our assumption  $A \subset \mathcal{L}$ , we have  $\mathcal{L}^* \subset A^*$ . Hence  $\lambda - A^*$  is injective and extends  $\lambda - \mathcal{L}^*$ . But  $(\lambda - \mathcal{L}^*)^{-1} = ((\lambda - \mathcal{L})^{-1})^*$  by Lemma 1.2, and it is continuous on  $Y = X^*$  w.r.t.  $\mathcal{C}(Y, X)$  by Lemma 1.5. Thus  $\lambda - \mathcal{L}^* : \mathbb{D}(\mathcal{L}^*) \rightarrow X^*$  is bijective. Consequently  $\lambda - A^* = \lambda - \mathcal{L}^*$ , the desired property (iii).

(iii)  $\Rightarrow$  (ii). Since  $\mathcal{L}$  is closed,  $A(\subset \mathcal{L})$  is closable and  $\bar{A} = A^{**} = \mathcal{L}^{**} = \mathcal{L}$  by Lemma 1.1.

(ii)  $\Rightarrow$  (i): trivial.

(2) The equivalence of (v), (v') with (i)  $\Leftrightarrow$  (iv).

(iii)  $\Rightarrow$  (v'). This is because  $(\lambda - A^*)^{-1} = (\lambda - \mathcal{L}^*)^{-1} = ((\lambda - \mathcal{L})^{-1})^*$  exists (and it is continuous on  $(Y, \mathcal{C}(Y, X))$ ).

(v')  $\Rightarrow$  (v) (for all  $\lambda > \lambda_0$ ): trivial.

(v) (for some  $\lambda > \lambda_0$ )  $\Rightarrow$  (iv). This is an immediate consequence of Hahn–Banach theorem.

(3) The equivalence of (vi) with (i)  $\Leftrightarrow$  (v).

(ii)  $\Rightarrow$  (vi) (following the classical proof, e.g., in Pazy [32]). Only the uniqueness requires a proof. Let  $v(\cdot)$  be a strong solution of (2.3) with  $v(0) = x \in \mathbb{D}(\bar{A}) = \mathbb{D}(\mathcal{L})$  (by (ii)). Fix an arbitrary  $a > 0$ , and consider  $h(t) := T_{a-t} v(t)$ .  $h(\cdot)$  is continuous from  $[0, a]$  to  $X_\beta$ . By the  $C_0$ -semigroup property of  $(T_t)$  and (2.3), for any  $t \in (0, a)$ ,

$$\begin{aligned} \frac{d}{dt} h(t) &= \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \left( [T(a-t-\varepsilon) - T(a-t)] v(t) + T(a-t-\varepsilon) [v(t+\varepsilon) - v(t)] \right) \\ &= -T_{a-t} \mathcal{L} v(t) + T_{a-t} \bar{A} v(t) = 0. \end{aligned}$$

Thus  $v(a) = h(a) = h(0) = T_a x$ , the desired (vi).

(vi)  $\Rightarrow$  (ii). Since  $\bar{A} \subset \mathcal{L}$ , for each  $x \in \mathbb{D}(\bar{A})$ , the strong solution  $v(t)$  of (2.3) is also a strong solution of (2.3) with  $\bar{A}$  substituted by  $\mathcal{L}$ . Since property (vi) always holds for  $\mathcal{L}$  instead of  $\bar{A}$ , we have  $v(t) = T_t x \in \mathbb{D}(\bar{A})$ .

Consequently  $\mathbb{D}(\bar{A})$  is stable by  $(T_t)_{t \geq 0}$ , and it is, recalling-it, dense in  $X_\beta$ , hence it is a core of  $\mathcal{L}$  by Lemma 2.4.

(4) *The equivalence of (vii), (vii') with (i)  $\Leftrightarrow$  (v).*

(vii)  $\Rightarrow$  (v). Assume in contrary that there were some  $y_0 \neq 0$  such that  $(\lambda - A^*)y_0 = 0$  for some  $\lambda > \lambda_0$ . We claim that for some  $t \geq 0$

$$T_t^* y_0 - e^{\lambda t} y_0 \neq 0.$$

Indeed otherwise

$$\left. \frac{d}{dt} T_t^* y_0 \right|_{t=0} = \left. \frac{d}{dt} e^{\lambda t} y_0 \right|_{t=0} = \lambda y_0$$

exists in  $(Y, \mathcal{C}(Y, X))$ . Then by Theorem 1.4,  $y_0 \in \mathbb{D}(\mathcal{L}^*)$  and  $(\lambda - \mathcal{L}^*)y_0 = 0$ . Thus  $y_0 = 0$ , a contradiction.

It is easy to see that  $(u(t) = T_t^* y_0 - e^{\lambda t} y_0)$  is a nonzero weak solution of Eq. (2.4) with initial condition  $u(0) = 0$ . This is in contradiction with (vii).

(ii)  $\Rightarrow$  (vii) (following closely to [42, Theorem 6.2]). It is enough to show that any  $\mathcal{C}(Y, X)$ -continuous weak solution  $u(t)$  of (2.4) with initial condition  $u(0) = 0$  must be zero for all  $t = a > 0$ . To show it, fix  $a > 0$  and  $x \in \mathcal{D}$ . Consider the function

$$h(t) := \langle x, T_t^* u(a - t) \rangle = \langle T_t x, u(a - t) \rangle.$$

We should show  $h(0) = h(a)$  (which yields  $\langle x, u(a) \rangle = \langle x, T_t^* u(0) \rangle = 0$  for any  $x \in \mathcal{D}$  which is dense in  $X_\beta$ , then  $u(a) = 0$ ).

Note at first that  $h(t)$  is continuous on  $[0, a]$ , for  $t \mapsto T_t x$  is continuous,  $\{T_t x; t \in [0, a]\}$  is compact in  $X_\beta$ , and  $t \mapsto u(a - t)$  is continuous from  $[0, a]$  to  $(Y, \mathcal{C}(Y, X))$ .

Thus for proving  $h(0) = h(a)$ , we have only to prove that

$$h'_+(t) := \lim_{\varepsilon \rightarrow 0+} \frac{h(t + \varepsilon) - h(t)}{\varepsilon} = 0, \quad \forall t \in (0, a) \quad (2.5)$$

by a well-known lemma in analysis [49, Chapter IX, p. 239]. Now write

$$\begin{aligned} h(t + \varepsilon) - h(t) &= \langle (T_{t+\varepsilon} - T_t)x, u(a - t - \varepsilon) \rangle + \langle T_t x, (u(a - t - \varepsilon) - u(a - t)) \rangle \\ &:= (I)_\varepsilon + (II)_\varepsilon. \end{aligned}$$

We begin with the first term  $(I)_\varepsilon$ . Since  $x \in \mathcal{D} \subset \mathbb{D}(\mathcal{L})$ ,  $T_t x \in \mathbb{D}(\mathcal{L})$  and  $\mathcal{L}T_t x = T_t \mathcal{L}x = T_t Ax$ . Consequently  $\{(T_{t+\varepsilon} - T_t)x/\varepsilon; 0 < \varepsilon \leq a - t\} \cup \{T_t Ax\}$  is compact in  $X_\beta$ . Since  $t \rightarrow u(t)$  is assumed to be continuous w.r.t.  $\mathcal{C}(Y, X)$ , thus

$$\lim_{\varepsilon \rightarrow 0+} \frac{(I)_\varepsilon}{\varepsilon} = \lim_{\varepsilon \rightarrow 0+} \frac{1}{\varepsilon} \langle (T_{t+\varepsilon} - T_t)x, u(a - t - \varepsilon) \rangle = \langle \mathcal{L}T_t x, u(a - t) \rangle. \quad (2.6)$$

To treat the second term  $(II)_\varepsilon$ , we require the key fact below:

$$\langle z, u(t) \rangle = \int_0^t \langle \mathcal{L}z, u(s) \rangle ds, \quad \forall z \in \mathbb{D}(\mathcal{L}), \quad t \geq 0. \quad (2.7)$$

Its proof is easy but essentially relies on (ii). At first (2.7) holds for  $z \in \mathcal{D}$  by (vii.2); next extend it to the whole  $\mathbb{D}(\mathcal{L})$  by the property (ii).

Applying (2.7) to  $z = T_t x$ , we obtain

$$\lim_{\varepsilon \rightarrow 0^+} \frac{(II)_\varepsilon}{\varepsilon} = - \lim_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon} \int_{a-t-\varepsilon}^{a-t} \langle \mathcal{L}T_t x, u(s) \rangle ds = - \langle \mathcal{L}T_t x, u(a-t) \rangle. \quad (2.8)$$

Combining (2.8) with (2.6), we get the desired (2.5).

(iii)  $\Rightarrow$  (vii)': trivial.

(vii)'  $\Rightarrow$  (v). Let  $(S_t)$  be the  $C_0$ -semigroup generated by  $A^*$  such that  $(e^{-at} S(t))$  is equicontinuous on  $(Y, \mathcal{C}(Y, X))$  for some constant  $a \in \mathbb{R}$ . Fix some  $\lambda > \max\{a, \lambda_0\}$ . If  $(Y, \mathcal{C}(Y, X))$  is sequentially complete, then  $\text{Ker}(\lambda - A^*) = \{0\}$ , for the resolvent  $(\lambda - A^*)^{-1}$  exists (and given by  $\int_0^\infty e^{-\lambda t} S_t y dt$ ). The argument below is to get around the sequential completeness assumption.

Let  $y_0 \in Y$  verify  $\lambda y_0 - A^* y_0 = 0$ . By the  $C_0$ -semigroup property,  $S_t y_0 \in \mathbb{D}(A^*)$  and  $\partial_t (e^{-\lambda t} S_t y_0) = e^{-\lambda t} S_t (-\lambda y_0 + A^* y_0) = 0$ . Consequently  $e^{-\lambda t} S_t y_0$  is a constant vector in  $Y$ , but it tends to 0 as  $t$  goes to infinity by our choice of  $\lambda$ . Hence  $y_0 = 0$ , the desired property (v).

(5) *The equivalence with (viii).*

*Necessity of (viii).* Let  $\mathcal{L}'$  be another extension of  $A$  such that  $\mathcal{L}'$  is the generator of some  $C_0$ -semigroup. By the equivalence of (i) and (ii), we have  $\mathcal{L}' = \bar{A} = \mathcal{L}$ .

*Sufficiency of (viii).* This is difficult and is proved in [24].  $\square$

## 2.2. A special case

Let now  $(Z, \|\cdot\|)$  be a Banach space and  $(X, \beta) = (Z^*, \mathcal{C}(Z^*, Z))$  (the setting of Corollary 1.7). Combining Corollary 1.7 and Theorem 2.1, we obtain:

**Theorem 2.5.** *Let  $(Z, \|\cdot\|)$  be a Banach space,  $X_\beta = (Z^*, \mathcal{C}(Z^*, Z))$  and  $A$  a linear operator on  $X$  with domain  $\mathcal{D}$  which is dense in  $X_\beta$ . Assume that there is a  $C_0$ -semigroup  $(T_t)$  on  $(Z, \|\cdot\|)$  with generator  $\mathcal{L}$  such that the dual  $\mathcal{L}^*$  extends  $A$ . Then the eight equivalent properties in Theorem 2.1 for  $A$  in  $X_\beta$  are equivalent to any one of the following:*

- (ix)  $A^*$  is the generator of some  $C_0$ -semigroup on  $(Z, \|\cdot\|)$ .
- (x) The closure of  $A$  w.r.t.  $\sigma(Z^*, Z)$  coincides with  $\mathcal{L}^*$ .
- (xi) For some (or equivalent for all)  $\lambda > \lambda_0$  where  $\lambda_0 := \lim_{t \rightarrow \infty} \frac{1}{t} \log \|T_t\|$ ,  $\text{Ker}(\lambda - A^*) = \{0\}$  in  $Z$ .

(xii) The dual Cauchy problem (2.4) has a unique  $\|\cdot\|$ -continuous solution  $t \rightarrow u(t) \in Z$  for any given initial condition  $z \in Z$ , i.e., satisfying

$$\langle u(t) - z, x \rangle = \int_0^t \langle u(s), Ax \rangle ds, \quad \forall x \in \mathcal{D}.$$

**Proof.** By Corollary 1.7,  $(T_t^*)$  is a  $C_0$ -semigroup on  $X_\beta = (Z^*, \mathcal{C}(Z^*, Z))$  with generator  $\mathcal{L}^*$ . Hence the existence assumption in Theorem 2.1 is satisfied.

Property (xii) is a translation of (vii) because  $\mathcal{C}(Z, X_\beta) = \|\cdot\|$ -topology of  $Z$ , by Lemma 1.10.

Property (xi) is a translation of (v).

Property (x) is equivalent to (ii) in Theorem 2.1, because the closure of the graph  $G(A)$  (which is a sub-vector space) in  $Z^* \times Z^*$  w.r.t.  $\mathcal{C}(Z^*, Z) \times \mathcal{C}(Z^*, Z)$  is the same as that w.r.t.  $\sigma(Z^*, Z) \times \sigma(Z^*, Z)$  (by Hahn–Banach theorem and the fact that the dual of  $Z^* \times Z^*$  endowed with these two locally convex topologies are the same, by Lemma 1.10).

Property (ix). It is a translation of (vii') in Theorem 2.1, for  $\mathcal{C}(Z, X_\beta)$  coincides with the  $\|\cdot\|$ -topology on  $Z$ .  $\square$

**Remark 2.6.** Let  $Z$  be a Banach space. If  $A^*$  is the generator  $\mathcal{L}_1$  of some  $C_0$ -semigroup  $(T_t^1)$  on  $Z$ , then  $A$  is essentially a generator on  $(Z^*, \mathcal{C}(Z^*, Z))$  (without the existence assumption in Theorem 2.5).

Indeed, assume that  $A^*$  is the generator  $\mathcal{L}_1$  of some  $C_0$ -semigroup  $(T_t^1)$  on the Banach space  $Z$ . By Corollary 1.7,  $\mathcal{L}_1^* = A^{**}$  is the generator of some  $C_0$ -semigroup on  $X_\beta$ . But by Lemma 1.1,  $A^{**}$  is the closure of  $A$  w.r.t.  $\sigma(Z^*, Z)$ , which coincides with its closure w.r.t.  $\mathcal{C}(X, Z)$ . Hence  $A$  is essentially a generator on  $(Z^*, \mathcal{C}(Z^*, Z))$ .

### 2.3. A Trotter–Kato approximation theorem

In Theorem 2.1, under the existence assumption, the main practical sufficient condition for the  $X_\beta$ -uniqueness of  $A$  is (v):  $\text{Ker}(\lambda - A^*) = \{0\}$  for some  $\lambda > 0$ . It is well known that the uniqueness is crucial for approximation. That is justified by the following version of the Trotter–Kato theorem.

**Theorem 2.7.** Let  $(T_t^{(n)})_{t \geq 0}$  be a sequence of  $C_0$ -semigroups with generators  $\mathcal{L}_n$  on  $X_\beta$  (sequentially complete) such that for some  $\lambda_0 \in \mathbb{R}$ ,

$$e^{-\lambda_0 t} T_t^{(n)}, \quad t \geq 0, n \in \mathbb{N}, \quad \text{are equicontinuous on } X_\beta. \quad (2.9)$$

Let  $A$  be a linear operator on  $X$  with domain  $\mathbb{D}(A) = \mathcal{D}$ . Assume

- (i)  $\mathcal{D}$  is a dense vector subspace in  $X_\beta$  and  $\mathcal{D} \subset \mathbb{D}(\mathcal{L}_n)$  for all  $n \in \mathbb{N}$ ;
- (ii) for all  $x \in \mathcal{D}$ , there are  $x_n \in \mathbb{D}(\mathcal{L}_n)$  such that  $\mathcal{L}_n x \rightarrow Ax$ ;
- (iii) for some  $\lambda > \lambda_0$ ,  $(\lambda - A)(\mathcal{D})$  is dense in  $X_\beta$  or equivalently  $\text{Ker}(\lambda - A^*) = \{0\}$ .

Then  $A$  is closable and its closure is the generator of some  $C_0$ -semigroup  $(T_t)$ . Moreover, for any  $x \in X$ ,

$$T_t^{(n)}x \rightarrow T_tx \quad \text{uniformly for } t \in [0, T], \quad \forall T > 0. \quad (2.10)$$

**Proof.** For any  $z = (\lambda - A)x$ , where  $x \in \mathcal{D}$ ,  $z_n := (\lambda - \mathcal{L}_n)x_n \rightarrow z$  by assumption (ii). By the equicontinuity assumption (2.9),  $(\lambda - \mathcal{L}_n)^{-1} = \int_0^\infty e^{-\lambda t} T_t^{(n)} dt$ ,  $n \in \mathbb{N}$ , are equicontinuous on  $X_\beta$  for any  $\lambda > \lambda_0$ . Thus

$$\lim_{n \rightarrow \infty} (\lambda - \mathcal{L}_n)^{-1}(z_n - z) = 0.$$

But  $(\lambda - \mathcal{L}_n)^{-1}z_n = x_n \rightarrow x$ . Therefore  $\lim_{n \rightarrow \infty} (\lambda - \mathcal{L}_n)^{-1}z = x$  in  $X_\beta$ . Since the set of such  $z \in (\lambda - A)(\mathcal{D})$  is dense in  $X_\beta$  by condition (iii), we have by the equicontinuity of  $\{(\lambda - \mathcal{L}_n)^{-1}, n \in \mathbb{N}\}$  and the sequential completeness of  $X_\beta$  that

$$\lim_{n \rightarrow \infty} (\lambda - \mathcal{L}_n)^{-1}z$$

exists for all  $z \in X$ , and this limit will be denoted by  $R_\lambda z$ . The range of  $R_\lambda$ , containing

$$R_\lambda(\lambda - A)(\mathcal{D}) = \mathcal{D}$$

(shown previously), is dense in  $X_\beta$ .

The facts established above allow us to apply the Trotter–Kato theorem [49, Chapter IX, Section 12, Theorem 1]: there exists some  $C_0$ -semigroup  $(T_t)$  such that the convergence (2.10) holds and  $R_\lambda$  coincides with the resolvent  $(\lambda - \mathcal{L})^{-1}$ , where  $\mathcal{L}$  is the generator of  $(T_t)$ .

Now for  $x \in \mathcal{D}$ ,  $R_\lambda(\lambda - A)x = x$  (shown above), hence  $x \in \mathbb{D}(\mathcal{L})$  and  $(\lambda - \mathcal{L})x = (\lambda - A)x$ , i.e.,  $A \subset \mathcal{L}$ . The existence assumption for  $A$  in Theorem 2.1 is then satisfied. By condition (iii) again and Theorem 2.1 (iv)  $\Rightarrow$  (ii), we have  $\bar{A} = \mathcal{L}$ .  $\square$

#### 2.4. Dependence of the uniqueness on topologies

**Theorem 2.8.** Let  $(T_t)$  be a  $C_0$ -semigroup on  $(X, \beta)$ . Let  $X_1$  be a dense vector subspace of  $X$ , but equipped with some locally convex and sequentially complete topology  $\beta_1$  which is stronger than  $\beta|_{X_1}$ . Assume that  $T_t X_1 \subset X_1$  and  $(T_t|_{X_1})$  is also a  $C_0$ -semigroup on  $(X_1, \beta_1)$ , whose generator is denoted by  $\mathcal{L}_1$ . Let  $\mathcal{D} \subset \mathbb{D}(\mathcal{L}_1)$  be a dense vector subspace of  $X_1$ .

If  $A = \mathcal{L}_1|_{\mathcal{D}}$  is essentially a generator on  $(X_1, \beta_1)$ , so it is on  $(X, \beta)$ .

**Proof.** Notice at first  $\mathcal{L}_1 \subset \mathcal{L}$ . Fix some  $\lambda_0 \geq 0$  such that  $(e^{-\lambda_0 t} T_t)$  is both equicontinuous on  $(X_1, \beta_1)$  and on  $(X, \beta)$ , and fix  $\lambda > \lambda_0$ . By the uniqueness of  $A$  in  $(X_1, \beta_1)$ , the  $\beta_1$ -closure of  $(\lambda - A)(\mathcal{D})$  coincides with  $X_1$  (by Theorem 2.1). But  $X_1$  is dense in  $(X, \beta)$ , then the  $\beta$ -closure of  $(\lambda - A)(\mathcal{D})$  is equal to  $X$ , which implies the desired result by Theorem 2.1.  $\square$

Theorem 2.8 above yields immediately

**Corollary 2.9.** Let  $(P_t)_{t \geq 0}$  be a sub-Markov  $C_0$ -semigroup on  $L^1(E, \mathcal{B}, \mu)$  where  $\mu$  is a probability measure. Then  $(P_t)$ , restricted to  $L^p(E, \mu)$ , is a  $C_0$ -semigroup on  $L^p(E, \mu)$  for all  $p \in (1, +\infty)$ , and it is so on  $(L^\infty(E, \mu), \mathcal{C}(L^\infty, L^1))$ .

Let  $1 \leq p < q \leq +\infty$  and  $A \in \mathcal{L}_{(p)}$  and  $A \in \mathcal{L}_{(q)}$  with  $\mathcal{D} = \mathbb{D}(A)$  dense in  $L^q$ , where  $\mathcal{L}_{(p)}$  is the generator of  $(P_t)$  in  $L^p(E, \mu)$ . If  $A$  is  $L^q(\mu)$ -unique (if  $q = +\infty$ ,  $L^\infty$  is equipped with the topology  $\mathcal{C}(L^\infty, L^1)$ ), then it is  $L^p(\mu)$ -unique.

### 3. Characterization of $C_0$ -semigroups on $L^\infty$

Since  $L^\infty = (L^1)^*$ , by Theorem 1.4 and Corollary 1.7, a suitable topology on  $L^\infty$  for studying  $C_0$ -semigroups is  $\mathcal{C}(L^\infty, L^1)$ . The main purpose of this section is to give a simplified Hille–Yosida theorem for sub-Markov  $C_0$ -semigroups on  $(L^\infty, \mathcal{C}(L^\infty, L^1))$ . See [7,15] for related works.

#### 3.1. Characterization of nonnegative $C_0$ -semigroups

Our purpose here is to find some much easier and concrete characterization than the Hille–Yosida condition (1.9) for sub-Markov  $C_0$ -semigroups on  $L^\infty$  endowed with the topology  $\mathcal{C}(L^\infty, L^1)$ .

Let  $E$  be a *Lusinian* topological space (i.e., homeomorphic to a Borel subset of some compact metrizable space  $\hat{E}$ ) and  $\mathcal{B}$  its Borel  $\sigma$ -field, and  $\mu$  a nonnegative (non-zero)  $\sigma$ -finite measure on  $(E, \mathcal{B})$ . A linear operator  $P$  on  $L^\infty(\mu) := L^\infty(E, \mathcal{B}, \mu)$  is said to be *nonnegative*, if for each  $f \geq 0$  ( $\mu$ -a.e.),  $Pf \geq 0$  ( $\mu$ -a.e.). It is called *sub-Markov*, if  $P$  is nonnegative and  $P_t 1 \leq 1$  ( $\mu$ -a.e.).

We first characterize the continuity of a nonnegative operator on  $L^\infty$  w.r.t.  $\mathcal{C}(L^\infty, L^1)$ .

**Lemma 3.1.** *Let  $P$  be a nonnegative and  $\|\cdot\|_\infty$ -bounded operator on  $L^\infty(E)$ . The following properties are equivalent:*

- (i)  $P$  is continuous w.r.t.  $\mathcal{C}(L^\infty, L^1)$  or  $\tau(L^\infty, L^1)$  or  $\sigma(L^\infty, L^1)$ .
- (ii) For any sequence  $(f_n) \subset L^\infty(\mu)$  decreasing  $\mu$ -a.e. to zero,  $Pf_n$  decreases to zero  $\mu$ -a.e. on  $E$ .
- (iii) There is a kernel realization  $\tilde{P}(x, dy)$  of  $P$ . More precisely,
  - (iii.a)  $\forall x \in E$ ,  $\tilde{P}(x, dy)$  is a measure on  $(E, \mathcal{B})$ ;
  - (iii.b)  $x \rightarrow \tilde{P}(x, B)$  is  $\mathcal{B}$ -measurable for each  $B \in \mathcal{B}$ ;  
(A real function on  $E \times \mathcal{B}$  satisfying (iii.a), (iii.b) is called a kernel.)
  - (iii.c) for each  $f \in b\mathcal{B}$  (the space of real bounded and  $\mathcal{B}$ -measurable functions on  $E$ ),

$$\tilde{P}f(x) := \int_E f(y) \tilde{P}(x, dy)$$

coincides with  $Pf$ ,  $\mu$ -a.e.

**Proof.** At first the continuities of  $P$  w.r.t. the three topologies in (i) are all equivalent by Lemma 1.12.

(i)  $\Rightarrow$  (ii). Let  $(f_n) \subset L^\infty(\mu)$  be a sequence decreasing to zero  $\mu$ -a.e. Then  $Pf_n$ , being also  $\mu$ -a.e. decreasing sequence of nonnegative functions, converges  $\mu$ -a.e. to some nonnegative function  $h$ .

On the other hand, let  $K$  be any compact subset of  $L^1(\mu)$ . Since

$$\langle g, f_n \rangle_\mu := \int_E g(x) f_n(x) d\mu(x) \rightarrow 0$$

for each  $g \in L^1(\mu)$ , and since  $g \rightarrow \langle g, f_n \rangle_\mu$ ,  $n \geq 0$ , are equicontinuous on  $(L^1(\mu), \|\cdot\|_1)$  (for  $|\langle g, f_n \rangle_\mu| \leq \|f_n\|_\infty \leq \|f_0\|_\infty \|g\|_1$ ), then we have by Ascoli–Arzelà theorem

$$\sup_{g \in K} |\langle g, f_n \rangle_\mu| \rightarrow 0.$$

In other words,  $f_n \rightarrow 0$  in the topology  $\mathcal{C}(L^\infty, L^1)$ . Hence  $Pf_n \rightarrow 0$  in topology  $\mathcal{C}(L^\infty, L^1)$ . Thus for any  $B \in \mathcal{B}$  with  $\mu(B) < +\infty$ ,

$$\int_B h d\mu = \lim_{n \rightarrow +\infty} \langle 1_B, P_t f_n \rangle_\mu = 0.$$

Whence  $h = 0$ ,  $\mu$ -a.e., the desired result.

(ii)  $\Rightarrow$  (iii). This is a famous result in probability, see Dellacherie, Meyer [12, Chapter IX, Section 11] (see also [12, Chapter V, Section 67]) for a much finer result.

(iii)  $\Rightarrow$  (i). The key consists to prove that adjoint operator  $P^{s*}$  maps  $L^1(\mu)$  to  $L^1(\mu)$ . In that case,  $P$  is continuous on  $L^\infty(\mu)$  w.r.t.  $\mathcal{C}(L^\infty, L^1)$ , by Lemma 1.12.

It is ready to check that

$$\mu \tilde{P}(B) := \int_E \mu(dx) \tilde{P}(x, B), \quad \forall B \in \mathcal{B},$$

is absolutely continuous w.r.t.  $\mu$ . Hence for any  $0 \leq g \in L^1(\mu)$

$$B \rightarrow \int_E g(x) P(x, B) \mu(dx)$$

is a measure absolutely continuous w.r.t.  $\mu$ , whose Radon–Nykodim density will be denoted by  $h$ . So we have for any  $0 \leq f \in L^\infty(\mu)$

$$\langle g, Pf \rangle_\mu = \int \mu(dx) g(x) \int f(y) \tilde{P}(x, dy) = \int_E h(y) f(y) \mu(dy),$$

where it follows that  $h \in L^1(\mu)$  and  $P^*g = h$ .

Finally for any  $g \in L^1(\mu)$ , we have  $P^*g = P^*g_+ - P^*g_- \in L^1(\mu)$ , which is exactly what we have to establish.  $\square$

**Lemma 3.2.** Let  $(P_\alpha)_{\alpha \in A}$  be a family of nonnegative bounded operators on  $L^\infty(E, \mu)$ . Then (a)  $\Rightarrow$  (b)  $\Leftrightarrow$  (c):

(a)  $(P_\alpha)_{\alpha \in A}$  is equicontinuous on  $(L^\infty, \mathcal{C}(L^\infty, L^1))$ ;

- (b)  $\sup_{\alpha \in A} \|P_\alpha\|_\infty < +\infty$  and for each sequence  $(f_n)_{n \in \mathbb{N}} \subset L^\infty$  decreasing pointwise to zero ( $\mu$ -a.e.),  $\text{ess sup}_{\alpha \in A} P_\alpha f_n$  (the least element in  $L^\infty(\mu)$  greater than  $P_\alpha f_n$ ,  $\mu$ -a.e. for every  $\alpha \in A$ ) decreases to zero ( $\mu$ -a.e.);
- (c)  $(P_\alpha)_{\alpha \in A}$  is equicontinuous on  $L^\infty(\mu)$  w.r.t. the Mackey topology  $\tau(L^\infty, L^1)$ .

In particular, if  $(Q_\alpha)_{\alpha \in A}$  is a family of nonnegative bounded operators on  $L^\infty(E, \mu)$ , equicontinuous w.r.t.  $\tau(L^\infty, L^1)$ , and such that for each  $\alpha \in A$ ,

$$P_\alpha f \leq Q_\alpha f, \quad \forall 0 \leq f \in L^\infty(\mu),$$

then  $(P_\alpha)_{\alpha \in A}$  is equicontinuous on  $L^\infty$  w.r.t.  $\tau(L^\infty, L^1)$ .

**Proof.** (a)  $\Rightarrow$  (b). The same proof as in Lemma 3.1.

(b)  $\Rightarrow$  (c). All properties in this lemma on  $L^\infty$  are the same w.r.t. a change of measure  $\tilde{\mu}$  equivalent to  $\mu$ . Choosing a probability measure equivalent to  $\mu$  if necessary, we can assume without loss of generality that  $\mu$  is itself a probability measure.

In this case, recall the well-known Dunford–Pettis theorem:

A bounded subset  $K$  in  $L^1(\mu)$  is  $\sigma(L^1, L^\infty)$ -relatively compact iff  $K$  is uniformly integrable, or iff

$$\sup_{g \in K} |\langle |g|, f_n \rangle_\mu| \rightarrow 0$$

for any sequence  $(f_n)_{n \in \mathbb{N}} \subset L^\infty$  decreasing pointwise to zero ( $\mu$ -a.e.).

From this classical result let us derive

**Claim.** Under condition (b), if  $K \subset L^1(\mu)$  is  $\sigma(L^1, L^\infty)$ -relatively compact, then  $\bigcup_{\alpha \in A} P_\alpha^*(K)$  is  $\sigma(L^1, L^\infty)$ -relatively compact.

Indeed,  $\bigcup_{\alpha \in A} P_\alpha^*(K)$  is bounded in  $L^1$  (easy), and for any sequence  $(f_n)_{n \in \mathbb{N}} \subset L^\infty$  decreasing pointwise to zero ( $\mu$ -a.e.), letting  $h_n := \text{ess sup}_{\alpha \in A} P_\alpha f_n$  which decreases to zero  $\mu$ -a.e.,

$$\sup_{\alpha \in A, g \in K} |\langle P_\alpha^* g, f_n \rangle_\mu| \leq \sup_{\alpha \in A, g \in K} \langle |g|, P_\alpha f_n \rangle_\mu \leq \sup_{g \in K} \langle |g|, h_n \rangle_\mu \rightarrow 0,$$

where the claim follows.

Now for any  $\sigma(L^1, L^\infty)$ -compact, convex and circled  $K$  in  $L^1$ , let  $K'$  be the convex, circled and closed hull of  $\bigcup_{\alpha \in A} P_\alpha^*(K)$ .  $K'$  is again  $\sigma(L^1, L^\infty)$ -compact by the claim above and Krein's theorem [34, Chapter IV, Theorem 11.4]. We then have for any  $f \in L^\infty(\mu)$ ,

$$\sup_{\alpha \in A, g \in K} |\langle g, P_\alpha f \rangle_\mu| = \sup_{\alpha \in A, g \in K} |\langle P_\alpha^* g, f \rangle_\mu| \leq \sup_{g \in K'} |\langle g, f \rangle_\mu|$$

which yields the desired equicontinuity of  $(P_\alpha)_{\alpha \in A}$  on  $L^\infty$  w.r.t. the Mackey topology  $\tau(L^\infty, L^1)$ .

(c)  $\Rightarrow$  (b). It is enough to show that if  $L^\infty(\mu) \ni f_n \downarrow 0$ ,  $\mu$ -a.e., then  $f_n \rightarrow 0$  in  $\tau(L^\infty, L^1)$ . For this purpose, let  $K$  be an arbitrary  $\sigma(L^1, L^\infty)$ -compact, convex and circled subset in  $L^1(\mu)$ .



Then  $\{ \|g\|; g \in K \}$  is  $\sigma(L^1, L^\infty)$ -relatively compact by Dunford–Petti’s theorem. Let  $\tilde{K}$  be the  $\sigma(L^1, L^\infty)$ -closure of  $\{ \|g\|; g \in K \}$ , which is  $\sigma(L^1, L^\infty)$ -compact. We have

$$\sup_{g \in \tilde{K}} |\langle g, f_n \rangle_\mu| \leq \sup_{g \in \tilde{K}} \langle g, f_n \rangle_\mu.$$

But since  $(g \rightarrow \langle g, f_n \rangle_\mu)$  is a sequence of  $\sigma(L^1, L^\infty)$ -continuous functionals on  $\tilde{K}$ , decreasing to 0, then it converges to zero uniformly on the  $\sigma(L^1, L^\infty)$ -compact  $\tilde{K}$  by Dini’s monotone convergence theorem. Consequently the last term in the above inequality decreases to zero, i.e.,  $f_n \rightarrow 0$  in  $\tau(L^\infty, L^1)$ .

Finally the last claim in this lemma follows from (b)  $\Rightarrow$  (c).  $\square$

**Proposition 3.3.** *Let  $(P_t)_{t \geq 0}$  be a semigroup of bounded nonnegative operators on  $L^\infty(\mu)$ . Then the following properties are equivalent:*

- (a)  $(P_t)_{t \geq 0}$  is a  $C_0$ -semigroup w.r.t.  $\mathcal{C}(L^\infty, L^1)$ ;
- (b)  $(P_t)_{t \geq 0}$  is a  $C_0$ -semigroup w.r.t.  $\tau(L^\infty, L^1)$ ;
- (c)  $P_t$  is a kernel for each  $t > 0$  and  $P_t f \rightarrow f$  as  $t \rightarrow 0$  in  $\sigma(L^\infty, L^1)$  for all  $f \in L^\infty$ .

**Proof.** (a)  $\Rightarrow$  (b). By (a),  $(e^{-\lambda_0 t} P_t)_{t \geq 0}$  is equicontinuous w.r.t.  $\mathcal{C}(L^\infty, L^1)$ , so w.r.t.  $\tau(L^\infty, L^1)$ , by Lemma 3.2. It remains to prove that for each  $f \in L^\infty$ ,  $t \rightarrow P_t f$  is continuous from  $\mathbb{R}^+$  to  $(L^\infty, \tau(L^\infty, L^1))$ . By the equicontinuity of  $(P_t)_{t \in [0, T]}$  (for each  $T > 0$ ) in  $\tau(L^\infty, L^1)$ , it is enough to prove that as  $t \rightarrow 0+$ ,

$$P_t f \rightarrow f \quad \text{w.r.t. } \tau(L^\infty, L^1), \quad \forall f \in \mathcal{D},$$

for some  $\mathcal{D}$  dense in  $(L^\infty, \tau(L^\infty, L^1))$ .

Let  $\mathcal{L}^C$  be the generator of  $(P_t)$  w.r.t.  $\mathcal{C}(L^\infty, L^1)$ , which is complete by Lemma 1.10. Choose  $\mathcal{D} := \mathbb{D}(\mathcal{L}^C)$ , which is dense in  $(L^\infty, \sigma(L^\infty, L^1))$ , then in  $(L^\infty, \tau(L^\infty, L^1))$  (by Hahn–Banach theorem). For each  $f \in \mathcal{D}$ ,

$$P_t f - f = \int_0^t P_s(\mathcal{L}^C f) ds$$

which converges to zero (as  $t \rightarrow 0+$ ) in the norm  $\|\cdot\|_\infty$ -topology, hence in the Mackey topology  $\tau(L^\infty, L^1)$ . This finishes the proof of (b).

(b)  $\Rightarrow$  (c). This is trivial by Lemma 3.1.

(c)  $\Rightarrow$  (a). It follows by Corollary 1.7.  $\square$

**Remark 3.4.** If one of the equivalent properties in this proposition holds, then the generators  $\mathcal{L}^C$  and  $\mathcal{L}^\tau$  of  $(P_t)$  w.r.t.  $\mathcal{C}(L^\infty, L^1)$  and  $\tau(L^\infty, L^1)$  coincide. Indeed  $f \in \mathbb{D}(\mathcal{L}^\tau)$  iff

$$P_t f - f = \int_0^t P_s(\mathcal{L}^\tau f) ds, \quad \forall t \geq 0,$$

iff  $f \in \mathbb{D}(\mathcal{L}^C)$  and  $\mathcal{L}^C f = \mathcal{L}^\tau f$ .

### 3.2. Hille–Yosida theorem for sub-Markov semigroups on $L^\infty$

**Theorem 3.5.** A linear operator  $\mathcal{L}$  on  $L^\infty(\mu)$  is the generator of some sub-Markov  $C_0$ -semigroup on  $(L^\infty(\mu), \mathcal{C}(L^\infty, L^1))$  or equivalently on  $(L^\infty(\mu), \tau(L^\infty, L^1))$ , if and only if the three conditions below are satisfied:

- (i)  $\mathcal{L}$  is a densely defined closed operator on  $L^\infty(\mu)$  w.r.t.  $\tau(L^\infty, L^1)$  (or  $\mathcal{C}(L^\infty, L^1)$  or  $\sigma(L^\infty, L^1)$  by Hahn–Banach);
- (ii) if  $\lambda > 0$ ,  $f \in \mathbb{D}(\mathcal{L})$ ,  $f - \lambda \mathcal{L}f = g$ , then

$$\operatorname{ess\,inf}_{x \in E} f(x) \geq \operatorname{ess\,inf}_{x \in E} g(x);$$

- (iii) the range of  $1 - \lambda \mathcal{L}$  is  $L^\infty(\mu)$  for any  $\lambda > 0$ .

Moreover, the closedness of  $\mathcal{L}$  w.r.t.  $\tau(L^\infty, L^1)$  in (i) can be replaced by

- (iv) for any sequence  $(f_n) \subset L^\infty(\mu)$  decreasing  $\mu$ -a.e. to zero,  $(1 - \lambda \mathcal{L})^{-1} f_n$  decreases to zero  $\mu$ -a.e. on  $E$ .

**Proof.** The necessity is easy (by Lemma 3.1 for the necessity of (iv)) and left to the reader. We prove now the sufficiency.

Applying (ii) to  $-f$ , we have

$$\|f\|_\infty \leq \|(1 - \lambda \mathcal{L})f\|_\infty.$$

Then by Theorem 1.15,  $\mathcal{L}$  is the generator of some  $C_0$ -semigroup  $(P_t)$  on  $(L^\infty(\mu), \mathcal{C}(L^\infty, L^1))$ , such that  $\|P_t\|_\infty \leq 1$ . It remains to show that  $P_t$  ( $t \geq 0$ ) is nonnegative for the sub-Markov property.

Since the resolvent  $(1 - \lambda \mathcal{L})^{-1}$  is nonnegative for any  $\lambda > 0$  (by (ii)), then for any  $0 \leq f \in L^\infty$ , by the proof of Hille–Yosida theorem [49, Chapter IX, Section 7], we have

$$\begin{aligned} P_t f &= \lim_{n \rightarrow \infty} \exp \left[ tn \left( \left( 1 - \frac{1}{n} \mathcal{L} \right)^{-1} - I \right) \right] f \\ &= \lim_{n \rightarrow \infty} e^{-nt} \sum_{k=0}^{\infty} \frac{(tn)^k}{k!} \left[ \left( 1 - \frac{1}{n} \mathcal{L} \right)^{-1} \right]^k f \\ &\geq 0 \quad (\mu\text{-a.e.}) \end{aligned}$$

(convergence in the topology  $\tau(L^\infty, L^1)$ ), i.e.,  $P_t$  is nonnegative.

Finally assume (iv) instead of the closedness of  $\mathcal{L}$  in  $(L^\infty, \tau(L^\infty, L^1))$ . By (ii) and (iii),  $(1 - \lambda \mathcal{L})^{-1}$  exists on  $L^\infty$  and it is nonnegative and its norm is  $\leq 1$ . So condition (iv) is meaningful and it implies  $(1 - \lambda \mathcal{L})^{-1}$  is continuous on  $(L^\infty(\mu), \mathcal{C}(L^\infty, L^1))$ , by Lemma 3.1. Thus  $\mathcal{L}$  is closed in  $(L^\infty(\mu), \mathcal{C}(L^\infty, L^1))$ . This completes the proof of this theorem.  $\square$

### 3.3. Proof of Theorem 0.2

**Proof.** This is a direct translation of Theorem 2.1.  $\square$

### 3.4. Laplacian on an open domain: capacity description

We now present two examples to illustrate the previous general results. Our objective here is pedagogic and especially for comparison of the  $L^\infty$ -uniqueness with the  $L^p$ -uniqueness. From now on  $L^\infty$  is equipped with the topology  $\mathcal{C}(L^\infty, L^1)$  of uniform convergence over compact subsets of  $L^1$  except explicit contrary statements.

In this section we are interested in the  $L^p(D, dx)$ -uniqueness of the Laplacian  $\Delta$  acting on the space  $C_0^\infty(D)$  of infinitely differentiable functions on  $D$  with compact support, where  $D$  is a nonempty open domain of  $\mathbb{R}^d$ . It is well known that  $(\Delta/2, C_0^\infty(D))$  is contained in the generator  $\Delta_{(p)}^D/2$  of the killed BM semigroup given by

$$P_t^D f(x) = \mathbb{E}^x 1_{[t < \sigma_D]} f(B_t), \quad (3.1)$$

where  $\sigma_D := \inf\{t \geq 0; B_t \notin D\}$  is the first exiting time of the BM from  $D$ .

For giving our answer to the uniqueness problem, we recall the important classical notion of capacity  $\text{Cap}_{r,p}$  associated with the two parameters  $1 < p < +\infty$  and  $r > 0$  (see [50]).

At first, for open  $O \subset \mathbb{R}^d$ ,

$$\text{Cap}_{r,p}(O) := \inf \left\{ \int_{\mathbb{R}^d} |(1 - \Delta)^{r/2} f|^p dx; f \in W^{r,p} \text{ and } f \geq 1_O, \text{ a.e. on } \mathbb{R}^d \right\}, \quad (3.2)$$

where  $W^{r,p}$  is the usual Sobolev space.

And next, for any subset  $A$  of  $\mathbb{R}^d$ ,

$$\text{Cap}_{r,p}(A) := \inf \{ \text{Cap}_{r,p}(O); O \text{ (open)} \supset A \}. \quad (3.3)$$

Let  $g_r$  be the Bessel potential, which is the integral kernel of  $(1 - \Delta)^{-r/2}$ . For  $r = 2$  which interested us here mostly, we have

$$g_2(x) := \int_0^\infty \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x|^2}{4t}\right) dt.$$

We recall also the following equivalence (see [50, Theorem 2.6.12]): for any Borel subset  $A$ ,

$$\frac{1}{[\text{Cap}_{r,p}(A)]^{1/p}} = \inf \{ \|g_r \star \mu\|_q; \mu \in M_1(A) \}, \quad (3.4)$$

where  $M_1(A)$  is the space of probability measures on  $\mathbb{R}^d$  with  $\mu(A^c) = 0$ , and  $g_r \star \mu$  is the usual convolution, and  $1/p + 1/q = 1$ .

We can now state

**Proposition 3.6.** *Let  $D$  be a nonempty open domain of  $\mathbb{R}^d$ .*

- (a) *Let  $p = 1$ .  $(\Delta, C_0^\infty(D))$  is  $L^1(D)$ -unique if and only if  $\text{Cap}_{1,2}(D^c) = 0$  (or equivalently the BM starting from any point of  $D$  cannot reach  $D^c$ ).*

- (b) Let  $1 < p \leq d/2$ . Then  $(\Delta, C_0^\infty(D))$  is  $L^p(D)$ -unique if and only if  $\text{Cap}_{2,p}(D^c) = 0$ .  
 (c) Let  $p \in (d/2, +\infty]$ . The Laplacian  $(\Delta, C_0^\infty(D))$  is  $L^p(D)$ -unique if and only if  $D^c = \emptyset$  or  $D = \mathbb{R}^d$ .  
 (d) In particular for  $D = \mathbb{R}^d \setminus \{o\}$ , the Laplacian  $(\Delta, C_0^\infty(D))$  is  $L^p(D)$ -unique if and only if  $1 \leq p \leq d/2$ .

**Proof.** Part (a) is contained in [42, Theorem 1.1].

(b) *Necessity* for  $1 < p < +\infty$ . If in contrary  $\text{Cap}_{2,p}(D^c) > 0$ , by (3.4) there is some probability measure  $\mu$  supported in  $D^c$  such that

$$g_2 \star \mu \in L^q(\mathbb{R}^d), \quad 1/p + 1/q = 1.$$

Note that  $(\Delta - 1)g_2 = \delta_0$  (the Dirac measure) in  $\mathcal{D}'(\mathbb{R}^d)$ , the space of Schwarz distributions on  $\mathbb{R}^d$ . Then  $(\Delta - 1)(g_2 \star \mu) = \mu$  in  $\mathcal{D}'(\mathbb{R}^d)$ , and hence  $(\Delta - 1)(g_2 \star \mu) = 0$  in  $\mathcal{D}'(D)$  because  $\mu$  is supported by  $D^c$ .

But by the assumed  $L^p$ -uniqueness of  $(\Delta, C_0^\infty(D))$  and Theorem 2.1(v),  $g_2 \star \mu$  must be zero. This contradiction shows the necessity in part (b).

*Sufficiency* in (b). Assume that  $\text{Cap}_{2,p}(D^c) = 0$ . Then  $D^c$  is of zero Lebesgue measure. Then the generator  $\Delta_{(p)}/2$  of the BM semigroup acting on the whole  $L^p(\mathbb{R}^d)$  is an extension of  $(\Delta/2, C_0^\infty(D))$ . Since the closure of  $(\Delta, C_0^\infty(\mathbb{R}^d))$  in  $L^p(\mathbb{R}^d)$  is  $\Delta_{(p)}$  by Theorem A in Section 3.5, it is enough to show that for each  $f \in C_0^\infty(\mathbb{R}^d)$  fixed, there is a sequence  $(f_n) \subset C_0^\infty(D)$  so that

$$f_n \rightarrow f, \quad \Delta f_n \rightarrow \Delta f, \quad \text{both in } L^p(\mathbb{R}^d).$$

Indeed, since  $\text{Cap}_{2,p}(D^c) = 0$ , there exists a sequence  $(\eta_n \in C_b^\infty(\mathbb{R}^d), n \in \mathbb{N})$  such that  $0 \leq \eta_n \leq 1$ ,  $\eta_n = 1$  over an open neighborhood of  $D^c$  and

$$\|\eta_n\|_p + \|\Delta \eta_n\|_p \rightarrow 0.$$

Set  $f_n := (1 - \eta_n)f \in C_0^\infty(D)$ . Obviously  $f_n \rightarrow f$  in  $L^p$ . And

$$\|\Delta f - \Delta f_n\|_p \leq \|\eta_n \Delta f\|_p + \|f \Delta \eta_n\|_p + 2\|\nabla f \cdot \nabla \eta_n\|_p,$$

where the last term is bounded from above by  $\|\nabla f\|_\infty C \|\Delta \eta_n\|_p$  (the boundedness of Riesz transformation). Hence  $\Delta f_n \rightarrow \Delta f$  in  $L^p$ , too.

(c) The sufficiency is contained in Theorem A in Section 3.5. We prove now the necessity.

For  $d/2 < p < +\infty$ , the necessity in (c) follows from the proof of the necessity in (b), because it is well known that  $\text{Cap}_{2,p}(A) = 0$  if and only if  $A$  is empty, see [50, Remark 2.6.15, p. 75].

In the case where  $p = +\infty$ , if in contrary  $D^c$  is not empty, say, contains  $x_0$ , then the Bessel potential  $g_2(x - x_0)$  is a  $L^1$ -integrable  $(\Delta - 1)$ -harmonic function on  $\mathbb{R}^d \setminus \{x_0\}$ , then on  $D$ . By Theorem 2.1,  $(\Delta, C_0^\infty(D))$  would be not unique on  $L^\infty$ , which contradicts with our necessity assumption.

(d) If  $p > d/2$ , this follows from (c). For  $p = 1$ , it follows from (a). For  $1 < p \leq d/2$  (necessarily  $d \geq 3$ ), by [50, Theorems 2.6.13 and 2.6.14],  $\text{Cap}_{2,p}(\{o\}) = 0$ . Thus the  $L^p$ -uniqueness of  $(\Delta, C_0^\infty(D))$  follows from (b).  $\square$

**Corollary 3.7.** *Let  $D$  be a nonempty open strict domain of  $\mathbb{R}^d$  and assume  $d \geq 2$ .*

- (a) *The  $L^\infty$ -Liouville property holds on  $D$ , i.e., any bounded  $(\Delta)$ -harmonic function in  $D$  is constant, if and only if  $\text{Cap}_{1,2}(D^c) = 0$ .*
- (b) *Let  $1 < p \leq d/2$ . If  $\text{Cap}_{2,p}(D^c) = 0$ , any  $L^q$ -integrable ( $1/p + 1/q = 1$ ) harmonic function in  $D$  is constant (zero, in fact).*

**Proof.** For (b) and the sufficiency in (a) both, notice that any harmonic function  $h$  in  $D$  which is in  $L^q(D) = L^q(\mathbb{R}^d)$  must be in the domain of the dual operator acting on  $L^q$  of  $(\Delta, C_0^\infty(D))$  acting on  $L^p(\mathbb{R}^d)$ . By the  $L^p$ -uniqueness of  $(\Delta, C_0^\infty(D))$  proven in Proposition 3.1 and by Theorem 2.1,  $h \in \mathbb{D}(\Delta_{(q)})$  and  $\Delta_{(q)}h = 0$  over the whole space  $\mathbb{R}^d$ . Thus the classical Liouville property on  $\mathbb{R}^d$  implies that  $h$  is constant.

For the necessity in (a), assume that  $\text{Cap}_{1,2}(D^c) > 0$  in contrary, i.e., the BM starting from any  $x \in D$  can reach  $\partial D$  with a positive probability. Then  $\text{Cap}_{1,2}(\partial D) > 0$ , too. Let us prove at first that  $h(x) := \mathbb{P}_x(\sigma_D < +\infty) = 1$  for all  $x \in D$ .

In fact,  $h(x)$  is a bounded harmonic function in  $D$ , hence  $h(x) = c$  on  $D$  by the assumed Liouville property. As  $h(x) > 0$  by the fact that  $\text{Cap}_{1,2}(D^c) > 0$ , then  $c > 0$ .

On the other hand,  $h(B_{t \wedge \sigma_D})$  is a continuous martingale (well known) and  $h(B_{\sigma_D}) = 1$  over  $[\sigma_D < +\infty]$ . Thus over  $[\sigma_D < +\infty]$  which is of positive  $\mathbb{P}_x$ -probability, we have

$$c = \lim_{t \uparrow \sigma_D} h(B_t) = h(B_{\sigma_D}) = 1,$$

the desired result.

Now having  $\mathbb{P}_x(\sigma_D < +\infty) = 1$  over  $D$ , we can apply two deep results for the so called fine Dirichlet problem (see [31, Theorem 9.14, p. 115])

At first, for any  $f \in C_b(\partial D)$ ,

$$\phi(x) := \mathbb{E}^x f(B_{\sigma_D}) \quad (3.5)$$

verifies (i)  $\Delta\phi(x) = 0$  in  $D$ , and (ii)  $\lim_{D \ni x \rightarrow y} \phi(x) = f(y)$  for any  $y \in \partial D$  which is ‘regular’ for  $D$ .

The second result crucial for us is that the set of the irregular points in  $\partial D$  is semipolar, or equivalently its  $\text{Cap}_{1,2}$  is zero. Since  $\text{Cap}_{1,2}(\partial D) > 0$ , the  $\text{Cap}_{1,2}$ -capacity of the set of regular points in  $\partial D$  is positive.

As  $d \geq 2$ , there are at least two different regular points  $y_0, y_1 \in \partial D$ . Take a bounded and continuous function  $f$  on  $\partial D$  so that  $f(y_0) \neq f(y_1)$ , we see that the bounded harmonic function on  $D$  given by (3.5) is not constant, a contradiction with the assumed Liouville property. The proof is completed.  $\square$

**Remark 3.8.** 1. Part (a) is a well-known result in harmonic analysis. The probabilistic proof above is essentially known to specialists.

2. Part (d) in Proposition 3.1 is known: see Eberle [14].

3. For  $1 < p \leq d/2$  ( $d/2$  is a well-known critical case in the Sobolev space theory), a very concrete condition for  $\text{Cap}_{2,p}(A) = 0$  is known, see [50, Theorem 2.6.16]: Let  $H^\alpha$  be the  $\alpha$ -dimensional outer Hausdorff measure. If  $H^{d-2p}(A) < +\infty$ ,  $\text{Cap}_{2,p}(A) = 0$ . Conversely if  $\text{Cap}_{2,p}(A) = 0$ , then  $H^{d-2p+\varepsilon}(A) = 0$  for any  $\varepsilon > 0$ .

Applying this criterion to a  $k$ -dimensional sub-manifold  $E$  of  $\mathbb{R}^d$ , we see that  $\text{Cap}_{2,p}(E) = 0$ , iff  $d - 2p \geq k$ .

4. Note that the necessary part in (a) is false in the one-dimensional case, for any bounded harmonic function on  $(0, +\infty)$  is constant but  $\text{Cap}_{1,2}(\{o\}) > 0$ .

### 3.5. The Laplacian $\Delta$ on a Riemannian manifold

Let  $M$  be a  $d$ -dimensional connected Riemannian manifold with volume measure  $dx$  and  $\Delta$  be the Laplace–Beltrami operator. There is a Brownian motion  $(B_t)$  with values in the one-point compactification space  $M \cup \partial$  defined on some filtered probabilities space  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, (\mathbb{P}_x)_{x \in M})$  with  $\mathbb{P}_x(B_0 = x) = 1$  for any initial point  $x \in M$ . Let  $e := \inf\{t \geq 0; B_t = \partial\}$  be the explosion time. When  $P_x(e = +\infty) = 1$  for all  $x \in M$ ,  $M$  is called *stochastically complete*. By Ito's formula  $(\Delta/2, C_0^\infty(M))$  is contained in the generator  $\Delta_{(p)}/2$  of the BM semigroup  $P_t f(x) := \mathbb{E}^x 1_{t < e} f(B_t)$  in  $L^p = L^p(M, dx)$  for all  $1 \leq p \leq +\infty$ .

**Theorem A.** (Essentially due to Yau [47,48] and Strichartz [38].) *If  $M$  is complete w.r.t. the Riemannian metric, then  $(\Delta/2, C_0^\infty(M))$  is  $L^p(M, dx)$ -unique for all  $1 < p < +\infty$ . In particular the generator  $\Delta_{(q)}/2$  of  $(P_t)$  in  $L^q(M, dx)$  for every  $1 < q < +\infty$  is given by*

$$\begin{aligned} \mathbb{D}(\Delta_{(q)}) &= \{f \in L^q(M, dx); \Delta f \in L^q(M, dx) \text{ (in distribution)}\}, \\ \Delta_{(q)} f &= \Delta f, \quad \forall f \in \mathbb{D}(\Delta_{(q)}). \end{aligned}$$

**“Proof.”** For the first claim, by Theorem 2.1, it is enough to show that if  $f \in (L^p(M, dx))^* = L^q(M, dx)$ , where  $1/p + 1/q = 1$  satisfies

$$\langle f, (1 - \Delta)h \rangle = 0, \quad \forall h \in C_0^\infty(M),$$

then  $f = 0$ . The condition above means simply  $(1 - \Delta)f = 0$  in the distribution sense. Then by Weyl's lemma,  $f$  admits a  $C^\infty$ -version. By Kato's inequality, we have in the distribution sense that

$$\Delta|f| \geq \text{sgn}(f)\Delta f = \text{sgn}(f)f = |f| \geq 0.$$

Thus  $|f|$  is subharmonic. Yau [47,48] prove that any  $L^q$  nonnegative subharmonic function must be constant on a complete Riemannian manifold. Hence  $|f|$  is constant, as well as  $f$  (for  $f$  admits a continuous version). Then  $f = \Delta f = 0$  as desired.

Since the operator described in the second claim is exactly  $(\Delta, \mathcal{D})^*$ , the second claim follows again from Theorem 2.1(iii).  $\square$

**Remark 3.9.** The uniqueness of  $C_0$ -semigroup in  $L^p$  generated by  $(\Delta, C_0^\infty(M))$  is proven by Strichartz [38]. Our Theorem 2.1 applied in this situation says that Yau's Liouville theorem implies both his uniqueness result and also the uniqueness of  $L^q$ -weak solutions of the heat equations  $\partial_t u = \Delta u$ .

**Theorem B.** (Davies [11, 1985]) *The following properties are equivalent:*

- (a)  $(\Delta, C_0^\infty(M))$  is  $L^1(M, dx)$ -unique;

- (b)  $M$  is stochastically complete;  
 (c) the generator  $\mathcal{L}_{(\infty)}$  of  $(P_t)$  in  $(L^\infty(M, dx), \mathcal{C}(L^\infty, L^1))$  is given by

$$\mathbb{D}(\Delta_{(\infty)}) = \{f \in L^\infty(M, dx); \Delta f \text{ (in the distribution sense)} \in L^\infty(M, dx)\},$$

$$\Delta_{(\infty)}f = \Delta f, \quad \forall f \in \mathbb{D}(\Delta_{(\infty)}).$$

Here the equivalence between (a) and (c) follows from Theorem 2.1.  
 What happens for  $p = +\infty$ ?

**Theorem C.** (P. Li [25, 1984]) Assume that  $M$  is a complete Riemannian manifold. If there are a point  $o \in M$  and some constant  $C > 0$  such that its Ricci curvature satisfies

$$\text{Ric}(x) \geq -C(1 + d(x, o)^2), \quad \forall x \in M, \quad (3.6)$$

then  $(\Delta, C_0^\infty(M))$  is  $L^\infty$ -unique, or equivalently the generator  $\Delta_{(1)}$  of  $(P_t)$  in  $L^1(M, dx)$  coincides with

$$\mathbb{D}(\Delta_{(1)}) = \{f \in L^1(M, dx); \Delta f \text{ (in the distribution sense)} \in L^1(M, dx)\},$$

$$\Delta_{(1)}f = \Delta f, \quad \forall f \in \mathbb{D}(\Delta_{(1)}).$$

**“Proof.”** At first the  $L^\infty$ -uniqueness of  $(\Delta, \mathcal{D})$  is equivalent to say that the dual  $(\Delta, \mathcal{D})^*$  acting on  $L^1$  is the generator of  $(P_t)$  in  $L^1$ , by Theorem 2.1. This shows the equivalence of the two conclusions in the theorem.

Under the curvature condition (3.6), P. Li [25] proved the Liouville theorem: if  $f$  is nonnegative, subharmonic and in  $L^1(M, dx)$ , then  $f$  is constant. Now the  $L^\infty$ -uniqueness of  $(\Delta, \mathcal{D})$  follows by the same argument as that of Theorem A.  $\square$

**Remark 3.10.** P. Li [25] also obtained the uniqueness of  $L^1$ -continuous weak solution of the heat equation  $\partial_t u = \Delta u$ , with several very refined techniques and gradient estimates. Theorem 2.1 says that his Liouville theorem implies in reality this last uniqueness. See X.D. Li [27] for recent study.

**Remark 3.11.** Li and Schoen [26] construct a finite volume complete and connected Riemannian manifold  $M$  (then stochastic complete) over which there is a non-constant  $L^1$ -harmonic function  $h_0$ . This might seem to be a little strange from the probabilistic point of view.

Indeed let  $\Delta_{(1)}$  be the generator of the Brownian semigroup  $(P_t)$  acting on  $L^1(M, dx)$ . If  $f \in \mathbb{D}(\Delta_{(1)})$  verifies  $\Delta_{(1)}f = f$ , then  $P_t f = f$ . Since  $(P_t)$  is ergodic and  $M$  is of finite volume,  $f = \int_M f dx / \text{vol}(M)$ , constant.

This apparently strange phenomena can be explained as follows: their  $L^1$ -harmonic function  $h_0$  (in the classical sense) is not in the domain  $\mathbb{D}(\Delta_{(1)})$ , and in particular  $\mathbb{D}(\Delta_{(1)}) \neq \{f \in L^1(M); \Delta f \in L^1(M) \text{ (distribution sense)}\}$  or equivalently  $(\Delta, C_0^\infty(M))$  is not unique in  $L^\infty$  (but  $L^1$ -unique by Davies Theorem C).

**Remark 3.12.** 1. A *sharp* sufficient condition for a complete Riemannian manifold to be stochastically complete is the following volume condition (due to Grigor'yan [18,19] and Karp, Li [21]: for some  $o \in M$ ,

$$\int_1^\infty \frac{r}{\log m(B(o, r))} dr = +\infty. \quad (3.7)$$

And there is a complete Riemannian manifold which is neither stochastically complete, nor satisfies condition (3.7) (due to Grigor'yan [18,19]).

We emphasize that the equivalence between the  $L^1$ -uniqueness and the conservativeness in Theorem B holds in a much more general setting: it is established for the regular symmetric elliptic operators by Davies [11], for singular generalized Schrödinger operators simultaneously by the first-named author [42–44] and Stannat [37].

2. Condition (3.6) is sharp for the  $L^\infty$ -uniqueness, or equivalently it is sharp for the uniqueness of the  $L^1$ -solution of the heat equation  $\partial_t u = \Delta u$  by Theorem 2.1. See Azencott [3] and Li, Schoen [26] for counter-examples and P. Li [25] for its explication. Remark that (3.6) implies (3.7), because under (3.6) there is some constant  $K > 0$  so that

$$m(B(o, r)) \leq K r^d \exp(r \sqrt{(d-1)C(1+r^2)}).$$

3. We also strongly recommend the reader to the works of Sturm [39,40] for the extensions to general Dirichlet forms, where the  $L^q$ -Liouville property for  $L - \lambda$  (here  $\mathcal{L}$  is a symmetric Markov generator,  $\lambda \geq 0$ ) is established under quite general conditions of Grigor'yan's type, but with the notable exception: the case where  $q = 1$  was not treated therein. Notice also a technical difference: in his definition  $(\mathcal{L} - \lambda)$ -generalized harmonic functions should be locally of finite energy (i.e., in  $H_{\text{loc}}^{1,2}$ ), is stronger than the distribution sense used in this paper. Some regularity results (such as Weyl's lemma type) are required for a weak  $(\mathcal{L} - \lambda)$ -harmonic function here becomes harmonic in his sense (this holds automatically for  $\mathcal{L} = \Delta$  on a Riemannian manifold). His very general results can neither be applied in the non-symmetric case, nor when his completeness assumption w.r.t. the Dirichlet form metric on the state space is not satisfied.

Summarizing those remarks, we see that the  $L^1$ - and  $L^\infty$ -uniqueness of  $(\Delta, C_0^\infty(M))$  (or equivalent the  $L^\infty$  and  $L^1$ -Liouville properties for  $\Delta - 1$ ) are quite different from (and much more difficult than) the  $L^p$ -uniqueness on a Riemannian manifold.

#### 4. Uniqueness of 1D diffusion operators

We now turn to the study of several important operators, which motivate further studies and help to understand the subtleness and the probabilistic meaning of  $L^\infty$ -uniqueness. Throughout this paper (except the explicit contrary statements),  $L^\infty$  will be endowed with the topology  $\mathcal{C}(L^\infty, L^1)$ , and the  $L^\infty$ -uniqueness of operators and  $C_0$ -semigroups, etc. on  $L^\infty$  are always w.r.t.  $\mathcal{C}(L^\infty, L^1)$ .

The purpose of this section is to study the  $L^\infty$ -uniqueness of one-dimensional diffusion generator. In a series of pioneering works [16,17], Feller investigated in a systematic and thorough way the different sub-Markov generator-extensions of the one-dimensional diffusion operator:

$$Af(x) = a(x)f'' + b(x)f', \quad \forall f \in C_0^\infty(x_0, y_0), \quad (4.1)$$



where  $-\infty \leq x_0 < y_0 \leq +\infty$ . Using the speed measure and scale function of Feller, Wiens [41] obtained the characterization of  $L^2$ -uniqueness (or equivalently the essential self-adjointness) of  $A$ . Furthermore, Eberle [14] and Djellout [13] have completely characterized the  $L^p$ -uniqueness of the one-dimensional regular diffusion operators for  $1 \leq p < \infty$ . The  $L^1$ -uniqueness is characterized by the first author [43]. The  $L^\infty$ -uniqueness is not studied before because of the lack of a natural definition of that uniqueness.

#### 4.1. $L^\infty$ -uniqueness

Assume that the coefficients  $a, b$  satisfy

$$a(x), b(x) \in L_{\text{loc}}^\infty(x_0, y_0; dx) \quad (4.2)$$

and the following very weak ellipticity condition:

$$a(x) > 0 \quad dx\text{-a.e.}; \quad \frac{1}{a(x)} \in L_{\text{loc}}^\infty(x_0, y_0; dx), \quad (4.3)$$

where  $L_{\text{loc}}^\infty(x_0, y_0; dx)$  (respectively  $L_{\text{loc}}^1(x_0, y_0; dx)$ ) denotes the space of real Lebesgue measurable functions which are essentially bounded (respectively integrable) w.r.t. Lebesgue measure on any compact sub-interval of  $(x_0, y_0)$ .

The corresponding stochastic differential equation is

$$dX_t = \sqrt{2a(X_t)} dB_t + b(X_t) dt. \quad (4.4)$$

It admits a martingale solution  $((X_t)_{0 \leq t < e}, (\mathbb{P}_x)_{x \in (x_0, y_0)})$  which is a Markov process, where  $e$  is the explosion time. Let  $(P_t)$  be its transition semigroup. By Ito's formula,  $A$  is contained in its generator  $\mathcal{L}_\infty$  in  $L^\infty((x_0, y_0), dx)$ .

Fix a point  $c \in (x_0, y_0)$  and let

$$s'(x) = \exp\left(-\int_c^x \frac{b(t)}{a(t)} dt\right), \quad m'(x) = \frac{1}{a(x)} \exp\left(\int_c^x \frac{b(t)}{a(t)} dt\right). \quad (4.5)$$

Their primitives  $s$  and  $m$  are respectively the scale and speed functions of Feller. Below  $m$  will also denote the measure  $m'(x) dx$  and it is easy to see that

$$\langle Af, g \rangle_m = \langle f, Ag \rangle, \quad \forall f, g \in C_0^\infty(x_0, y_0),$$

where  $\langle f, g \rangle_m := \int_{x_0}^{y_0} f(x)g(x)m'(x) dx$ . For  $h \in C_0^\infty(x_0, y_0)$ , we can write  $A$  in Feller's form

$$Af = \frac{f''}{m's'} + \frac{1}{m'} \left(\frac{1}{s'}\right)' f' = \frac{1}{m'} \left(\frac{f'}{s'}\right)' = \frac{d}{dm} \frac{d}{ds} f.$$

Now regard  $(A, C_0^\infty(x_0, y_0))$  as an operator on  $L^\infty(x_0, y_0; dx) = L^\infty(x_0, y_0; m'(x) dx) =: L^\infty(m)$ , which is endowed with the topology  $\mathcal{C}(L^\infty(m), L^1(m))$ . Our purpose is to find an explicit characterization of the  $L^\infty$ -uniqueness of  $(A, C_0^\infty(x_0, y_0))$ . To this end let us recall the following important notion of “no entrance boundary” of Feller.

**Definition 4.1.** We call that  $y_0$  is a *no entrance boundary* if

$$\int_c^{y_0} m'(y) dy \left\{ \int_c^y s'(x) dx \right\} = +\infty; \quad (4.6)$$

$x_0$  is a *no entrance boundary* if

$$\int_{x_0}^c m'(y) dy \left\{ \int_y^c s'(x) dx \right\} = +\infty. \quad (4.7)$$

One main result of this section is

**Theorem 4.1.**  $(A, C_0^\infty(x_0, y_0))$  is unique in  $L^\infty(m)$  iff both  $x_0$  and  $y_0$  are no entrance boundary (i.e., both (4.6) and (4.7) hold).

As all weakly  $A$ -harmonic functions (i.e.,  $Ah = 0$ , a.e.) can be solved explicitly as  $c_1 s(x) + c_2$ , the necessary and sufficient condition above means simply that  $s(x)$  is neither in  $L^1([c, y_0], m)$  nor in  $L^1((x_0, c], m)$ . The reader might guess immediately that no entrance boundary is also closely related with the  $L^1(m)$ -Liouville property, which will be the object of Theorem 4.13.

**Remark 4.2.** In the classification of Feller,  $y_0$  is called *no accessible boundary*, if

$$\int_c^{y_0} s'(y) dy \int_c^y m'(x) dx = +\infty. \quad (4.8)$$

Similar definition for  $x_0$ .

The  $L^1(m)$ -uniqueness of  $A$  is known to be equivalent to the conservativeness of the diffusion (4.4) by the first author [43] or Eberle [14], i.e.,  $x_0$  and  $y_0$  are both no accessible. This means that a particle with starting point inside  $(x_0, y_0)$  cannot reach the boundary. However,  $L^\infty(m)$ -uniqueness of  $A$  means, by the proposition above, that a particle “starting from the boundary” cannot enter in  $(x_0, y_0)$ .

**Remark 4.3.** Eberle [14] and Djellout [13] proved that  $A$  is  $L^p(m)$ -unique iff  $s(x)$  is neither in  $L^q([c, y_0], m)$  nor in  $L^q((x_0, c], m)$ , where  $1/p + 1/q = 1$ .

**Remark 4.4.** A fairly good nature of the  $L^\infty$ -uniqueness is its independence to the reference measure  $m$ : the uniqueness of  $A$  in  $L^\infty(m)$  and that in  $L^\infty(\mu)$  are equivalent for any  $\sigma$ -finite nonnegative measure  $\mu \sim m$ .

For instance let  $\mu = \int_0^\infty e^{-t} \nu P_t dt$  where  $\nu$  is any probability measure on  $(x_0, y_0)$ . Assume that  $a(x)$  is locally bounded from below by a positive constant (locally uniform ellipticity) and  $A$  is  $L^\infty(m)$ -unique. In such case, As  $P_t(x, dy) \sim dy \sim m$ ,  $A$  is  $L^\infty(\mu)$ -unique, then  $L^p(\mu)$ -unique for all  $p \in [1, +\infty)$  by Corollary 2.9.

We begin with a series of lemmas.

**Lemma 4.5.** Let  $A^*: \mathbb{D}(A^*) \subset L^1(m) \rightarrow L^1(m)$  be the dual operator of  $A$  and  $u \in L^1(m)$ . Then  $u \in D(A^*)$  if and only if:

- (i)  $u$  has an absolutely continuous  $dx$ -version  $\tilde{u}$  such that  $\tilde{u}'$  is absolutely continuous;
- (ii)  $g := a\tilde{u}'' + b\tilde{u}' = (1/m')(\tilde{u}'/s')' \in L^1(m)$ .

In that case  $A^*u = g$ .

**Proof.** The sufficiency follows easily by integration by parts. Below we prove the necessity. Let  $x_0 < x_1 < y_1 < y_0$ . The space of distributions on  $(x_1, y_1)$  is denoted by  $\mathcal{D}'(x_1, y_1)$ .

(i) We recall that if  $k \geq 1$  and  $T_1, T_2 \in \mathcal{D}'(x_1, y_1)$  satisfy  $T_1^{(k)} = T_2^{(k)}$  (i.e.,  $\langle T_1, f^{(k)} \rangle = \langle T_2, f^{(k)} \rangle \forall f \in C_0^\infty(x_1, y_1)$ ) then there exists a polynomial  $v$  such that  $T_1 = T_2 + v$ .

(ii) Let  $u \in L^1(m)$  be in  $D(A^*)$  such that  $A^*u = g \in L^1(m)$ . Since  $u/a \in L_{\text{loc}}^1((x_0, y_0), dx)$ ,  $u, u/s'$ , then  $u(1/s')' \in L_{\text{loc}}^1((x_0, y_0), dx)$ . For  $f \in C_0^\infty(x_1, y_1)$ , we have

$$\int_{x_1}^{y_1} u(f'/s')' dx = \langle u, Af \rangle_m = \langle A^*u, f \rangle_m = \langle g, f \rangle_m = \int_{x_1}^{y_1} fgm' dx. \quad (4.9)$$

In particular

$$\begin{aligned} \left| \int_{x_1}^{y_1} u((1/s')f'' + (1/s')'f') dx \right| &= \left| \int_{x_1}^{y_1} fgm' dx \right| \\ &\leq \|gm'\|_{L^1(x_0, y_0; dx)} \cdot \|f\|_{L^\infty(x_1, y_1; dx)} \\ &\leq C \|f'\|_{L^1(x_1, y_1; dx)}, \end{aligned}$$

where  $C = \|gm'\|_{L^1(x_0, y_0; dx)}$  is independent of  $f$ . The above inequality means that the linear functional

$$l_u(\eta) := \int_{x_1}^{y_1} u((1/s')\eta' + (1/s')'\eta) dx, \quad \eta \in \{f'; f \in C_0^\infty(x_1, y_1)\} \subset L^1(x_1, y_1; dx)$$

is continuous w.r.t. the  $L^1(x_1, y_1; dx)$ -norm. Thus by the Hahn–Banach's theorem and the fact that the dual of  $L^1(x_1, y_1; dx)$  is  $L^\infty(x_1, y_1; dx)$ , there exists  $v \in L^\infty(x_1, y_1; dx)$  such that

$$l_u(\eta) = \int_{x_1}^{y_1} u((1/s')\eta' + (1/s')'\eta) dx = \int_{x_1}^{y_1} v\eta dx \quad (\eta \in \{f'; f \in C_0^\infty(x_1, y_1)\})$$

which implies

$$\int_{x_1}^{y_1} u(1/s')\eta' dx = \int_{x_1}^{y_1} (v - u(1/s')')\eta dx = \int_{x_1}^{y_1} h\eta' dx \quad (\eta \in \{f'; f \in C_0^\infty(x_1, y_1)\}),$$

where  $h(x) = -\int_{x_1}^x (v(t) - u(t)(1/s')'(t)) dt$  is an absolutely continuous function on  $(x_1, y_1)$ . It follows from (i) that there exists a polynomial  $w$  such that  $u(1/s') = h + w$  on  $(x_1, y_1)$  in the sense of distribution, hence  $u(1/s') = h + w$  a.e. on  $(x_1, y_1)$ .

(iii) Since  $1/s' > 0$  is absolutely continuous, the equality  $u = s'(h + w)$  (a.e.) shows that  $u$  also has an absolutely continuous version  $\tilde{u} := s'(h + w)$ .

(iv) Now we have by (4.9)

$$\int_{x_1}^{y_0} f g m' dx = \int_{x_1}^{y_1} \tilde{u} ((1/s') f')' dx = - \int_{x_1}^{y_1} \tilde{u}' (1/s') f' dx.$$

Hence  $((1/s')\tilde{u}')' = g m' \in L^1((x_1, y_1), dx)$  in the sense of distribution, then  $(1/s')\tilde{u}'$  has an absolutely continuous version, so is  $\tilde{u}'$ . Consequently  $g = (1/m')((1/s')\tilde{u}')' = a\tilde{u}'' + b\tilde{u}'$ , a.e. on  $(x_1, y_1)$ .

The lemma is thus proved since  $(x_1, y_1)$  is an arbitrary relatively compact subinterval of  $(x_0, y_0)$ .  $\square$

**Lemma 4.6.** Let  $h \in L^1(m)$  satisfies

$$(I - A^*)h = 0 \quad \text{that is,} \quad ((1/s')h')' = m'h \quad (4.10)$$

in the sense of Lemma 4.5. We may suppose that  $h$  is  $C^1$  such that  $h'$  is absolutely continuous.

Suppose that  $c_1 \in (x_0, y_0)$ ,  $h(c_1) > 0$  and  $h'(c_1) > 0$  (respectively  $< 0$ ). Then  $h'(y) > 0$  (respectively  $< 0$ ) for  $\forall y \in (c_1, y_0)$  (respectively  $\forall y \in (x_0, c_1)$ ).

**Proof.** Suppose  $h'(c_1) > 0$ . Let

$$\hat{y} = \sup\{y \geq c_1; h'(z) > 0 \forall z \in [c_1, y_0]\}. \quad (4.11)$$

It is clear that  $\hat{y} > c_1$ . Then  $h(t) \geq h(c_1) > 0 \forall t \in [c_1, \hat{y}]$ . By (4.10) we have for  $y \in (c_1, y_0)$

$$(1/s'(y))h'(y) - (1/s'(c_1))h'(c_1) = \int_{c_1}^y m'(t)h(t) dt. \quad (4.12)$$

If  $\hat{y} < y_0$ , (4.12) implies that

$$(1/s'(\hat{y}))h'(\hat{y}) = (1/s'(c_1))h'(c_1) + \int_{c_1}^{\hat{y}} m'(t)h(t) dt > (1/s'(c_1))h'(c_1) > 0.$$

It follows that  $h'(\hat{y}) > 0$ , hence  $h' > 0$  on  $[\hat{y}, \hat{y} + \varepsilon]$  for small  $\varepsilon$  which contradicts (4.11).

In the same way one can prove that if  $h'(c_1) < 0$ , then  $h'(x) < 0$  for all  $x \in (x_0, c_1)$ .  $\square$

**Lemma 4.7.** *There exist two strictly positive  $C^1$ -functions  $h_k$ ,  $k = 1, 2$  on  $(x_0, y_0)$  such that:*

- (1) *for  $k = 1, 2$ ,  $h'_k$  is absolutely continuous, and  $((1/s')h'_k)' = m'h_k$ , a.e.;*
- (2)  *$h'_1 > 0$  and  $h'_2 < 0$  over  $(x_0, y_0)$ .*

**Proof.**  $h_2$  was constructed by Feller [16, Lemma 9.1] in the case where  $a = 1$ . But his proof works in the actual general framework, so we indicate only the idea of Feller. Letting  $x_\delta \in (x_0, c)$  decrease to  $x_0$  and  $y_\delta \in (c, y_0)$  increase to  $y_0$ , as  $\delta$  decreases to zero. Consider the solution  $h_{1,\delta}$  of  $((1/s')h')' = m'h$  in the sense above joining  $(x_\delta, 0)$  to  $(c, 1)$ , and the solution  $h_{2,\delta}$  of the equation joining  $(c, 1)$  to  $(y_\delta, 0)$  (their existence and uniqueness follow by the classical ODE theory). Then  $h_1 = \lim_{\delta \rightarrow 0} h_{1,\delta}$  and  $h_2 = \lim_{\delta \rightarrow 0} h_{2,\delta}$ .  $\square$

**Proof of Theorem 4.1.** By the adjoint criterion in Theorem 2.1, we have only to show that Eq. (4.10) has no non-trivial  $L^1(m)$  solution iff (4.6) and (4.7) hold.

**Part “if.”** Assume (4.6) and (4.7) hold. Suppose in contrary that  $h \in L^1(m)$  and  $h \not\equiv 0$  satisfies (4.10). We can assume that  $h \in C^1(x_0, y_0)$  and  $h > 0$  on some interval  $[x_1, y_1] \subset (x_0, y_0)$  where  $x_1 < y_1$ . We notice that by (4.10),  $h' \not\equiv 0$  on  $(x_1, y_1)$ .

*Case (i).*  $h'(c_1) > 0$  for some  $c_1 \in (x_1, y_1)$ . By Lemma 4.6,  $h(y) \geq h(c_1) > 0$  for  $y \in [c_1, y_0]$ . We have by (4.12)

$$\begin{aligned} h(y) &= h(c_1) + \int_{c_1}^y h'(x) dx = h(c_1) + \int_{c_1}^y \left\{ \frac{h'(c_1)}{s'(c_1)} s'(x) + s'(x) \int_{c_1}^x m'(t) h(t) dt \right\} dx \\ &> \frac{h'(c_1)}{s'(c_1)} \int_{c_1}^y s'(x) dx. \end{aligned}$$

Thus by the fact that  $y_0$  is no entrance boundary,

$$\int_{c_1}^{y_0} h(y) m'(y) dy \geq \frac{h'(c_1)}{s'(c_1)} \int_{c_1}^{y_0} m'(y) dy \left\{ \int_{c_1}^y s'(x) dx \right\} = +\infty$$

a contradiction with the assumption  $h \in L^1(m)$ .

*Case (ii).*  $h'(c_1) < 0$  for some  $c_1 \in (x_1, y_1)$ . We prove in a similar way that  $\int_{x_0}^{c_1} m'(y) h(y) dy = +\infty$ .

**Part “only if.”** Assume that (4.7) (similar in the case (4.6): use  $h_1$  in Lemma 4.7 in the proof below) does not hold, that is,

$$\int_{x_0}^c m'(y) dy \left\{ \int_y^c s'(t) dt \right\} < +\infty.$$

Notice that we have in particular  $\int_{x_0}^c m'(y) dy < \infty$ . Below we shall prove that  $h = h_2$  in Lemma 4.7 is an element of  $L^1(m)$ . Recall that  $h > 0$  and  $h' < 0$  over  $(x_0, y_0)$ .

(1) *Integrability near  $y_0$ .* Let  $c \in (x_0, y_0)$ . For  $y \in (c, y_0)$  we have

$$0 \geq h'(y)/s'(y) = h'(c)/s'(c) + \int_c^y m'(t)h(t) dt$$

which implies that  $\int_c^{y_0} m'(t)h(t) dt \leq -h'(c)/s'(c) < +\infty$ .

(2) *Integrability near  $x_0$ .* Let  $c_0 \in (x_0, c)$ , since for  $x < c_0$

$$\int_x^c m'(s)h(s) ds > \int_{c_0}^c m'(t)h(t) dt > 0,$$

hence there exists  $\lambda > 1$  such that

$$(1/s'(c))h'(c) > -(\lambda - 1) \int_x^c m'(t)h(t) dt, \quad \forall x \in (x_0, c_0).$$

Then for  $x < c_0$

$$(1/s'(x))h'(x) = (1/s'(c))h'(c) - \int_x^c m'(t)h(t) dt \geq -\lambda \int_x^c m'(t)h(t) dt,$$

where it follows that for all  $y < c_0$ ,

$$h(y) = h(c_0) - \int_y^{c_0} h'(x) dx \leq h(c_0) + \lambda \int_y^{c_0} s'(x) \left[ \int_x^c m'(t)h(t) dt \right] dx.$$

Thus

$$\begin{aligned} m'(y)h(y) &\leq h(c_0)m'(y) + \lambda m'(y) \int_y^{c_0} s'(x) \left[ \int_x^c m'(t)h(t) dt \right] dx \\ &\leq h(c_0)m'(y) + \lambda \left( \int_{c_0}^c m'(t)h(t) dt \right) m'(y) \int_y^{c_0} s'(t) dt \\ &\quad + \lambda m'(y) \int_y^{c_0} s'(x) \left[ \int_x^{c_0} m'(t)h(t) dt \right] dx. \end{aligned}$$

Denoting  $\varphi(z) = \int_z^{c_0} m'(y)h(y) dy$  and putting

$$K = h(c_0) \int_{x_0}^{c_0} m'(y) dy + \lambda \int_{c_0}^c m'(t)h(t) dt \cdot \int_{x_0}^{c_0} \left\{ m'(y) \left[ \int_y^{c_0} s'(t) dt \right] \right\} dy < +\infty$$

the above inequality implies that for  $z < c_0$

$$\begin{aligned} \varphi(z) &\leq K + \lambda \int_z^{c_0} m'(y) \left[ \int_y^{c_0} s'(x)\varphi(x) dx \right] dy \\ &\leq K + \lambda \int_z^{c_0} m'(y) \left[ \int_y^{c_0} s'(x) dx \right] \varphi(y) dy. \end{aligned}$$

It follows from the Gronwall's inequality that

$$\varphi(z) \leq K \exp \left\{ \lambda \int_z^{c_0} m'(y) \left[ \int_y^{c_0} s'(x) dx \right] dy \right\}.$$

This shows that  $\varphi(x_0) < \infty$ , the  $m$ -integrability of  $h$  near  $x_0$ .  $\square$

**Corollary 4.8** (Comparison principle). *Let*

$$A_k f(x) = a_k(x)f''(x) + b_k(x)f', \quad \forall f \in C_0^\infty(x_0, y_0),$$

where  $(a_k, b_k)$ ,  $k = 1, 2$  satisfy (4.2) and (4.3). Assume that for some  $\delta > 0$ ,

$$\operatorname{sgn}(x - c) \frac{b_1(x)}{a_1(x)} \leq \operatorname{sgn}(x - c) \frac{b_2(x)}{a_2(x)} \quad \text{and} \quad a_1(x) \geq \delta a_2(x) \quad (4.13)$$

for all  $x$  sufficiently close to  $\{x_0, y_0\}$ . If  $A_1$  is  $L^\infty(m)$ -unique, so is  $A_2$ .

When  $a_1 = a_2$ , condition (4.13) means that the diffusion generated by  $A_2$  has a larger drift force to the direction of the boundaries, so intuitively if  $x_0, y_0$  are no entrance boundary for  $A_1$ , they will be so for  $A_2$ .

**Proof.** Let  $(s'_k, m'_k)$  be the derivatives of the scale and speed functions of  $A_k$ . This corollary follows from Theorem 4.1 and the equality

$$\int_c^{y_0} m'_k(y) dy \int_c^y s'_k(x) dx = \int_c^{y_0} \frac{1}{a_k(y)} dy \int_c^y \exp \left( \int_x^y \frac{b_k(t)}{a_k(t)} dt \right) dx$$

and a similar equality related with  $x_0$ .  $\square$

To apply the comparison principle above, we now check standard examples for no entrance boundary condition.

**Corollary 4.9.** *Let  $Af = f'' - \beta f'$ , where  $\beta: [0, +\infty) \rightarrow [1, +\infty)$  is a non-decreasing function. Then  $+\infty$  is a no entrance boundary for  $A$  if and only if*

$$\int_0^{\infty} \frac{1}{\beta(x)} dx = +\infty. \quad (4.14)$$

**Proof.** Let  $\phi(x) = \int_0^x \beta(t) dt$ . Then  $s'(x) = e^{\phi(x)}$  and  $m'(x) = e^{-\phi(x)}$

$$\int_0^{\infty} m'(x) dx \int_0^x s'(t) dt = \int_0^{\infty} \frac{\int_0^x e^{\phi(t)} dt}{e^{\phi(x)}} dx.$$

Now the desired result follows from

$$e^{\phi(x)} = e^{\phi(0)} + \int_0^x \beta(t) e^{\phi(t)} dt \leq e^{\phi(0)} + \beta(x) \int_0^x e^{\phi(t)} dt$$

and

$$e^{\phi(x)} \geq \beta(x/2) \int_{x/2}^x e^{\phi(t)} dt \geq \frac{\beta(x/2)}{2} \int_0^x e^{\phi(t)} dt. \quad \square$$

Let us give several examples to illustrate the differences of  $L^\infty$ -uniqueness and  $L^p$ -uniqueness.

**Example 4.10.** Let  $(x_0, y_0) = \mathbb{R}$ ,  $Af = f'' - \phi'(x)f'$  for all  $f \in C_0^\infty(\mathbb{R})$ , where  $\phi$  is locally Lipschitzian on  $\mathbb{R}$ . For this model,

$$s'(x) = e^{\phi(x)}, \quad m'(x) = e^{-\phi(x)}.$$

In such case  $A$  is  $L^p(m)$ -unique for every  $1 < p < +\infty$  by Eberle [14].

(1) When  $\phi_\alpha(x) = c|x|^\alpha$  for all  $|x|$  large enough where  $c > 0$ , by Corollary 4.9,  $+\infty$  is no entrance for  $A_\alpha f = f'' - \phi'_\alpha f'$  iff  $\alpha \leq 2$ , and the same holds for  $-\infty$  by symmetry. Hence  $A_\alpha f = f'' - \phi'_\alpha f'$ ,  $f \in C_0^\infty(\mathbb{R})$  is  $L^\infty(m)$ -unique iff  $\alpha \leq 2$ .

(2) Using the comparison principle in Corollary 4.8 and the example (1) above, we have immediately the following.

Let  $\phi$  is pair. If

$$\phi'(x) \geq cx^{\alpha-1}, \quad c > 0,$$



for all  $x \geq L > 0$  and  $\alpha > 2$ ,  $A$  is not  $L^\infty(m)$ -unique. In contrary, if

$$\phi'(x) \leq cx^{\alpha-1}, \quad c > 0,$$

for all  $x \geq L > 0$  and  $\alpha \leq 2$ , then  $A$  is  $L^\infty(m)$ -unique. In particular, if

$$c_1x^{\alpha-1} \leq \phi'(x) \leq c_2x^{\alpha-1}$$

for all  $x \geq L > 0$  where  $c_2 > c_1 > 0$ , then  $A$  is  $L^\infty(m)$ -unique iff  $\alpha \leq 2$ .

(3) Assume now that  $\phi \in C^2(\mathbb{R})$  and the Bakry–Emery curvature  $\phi''(x)$  associated with  $A$  (see [4] for a systematic development) verifies

$$\phi''(x) \geq \delta|x|^{\alpha-2}$$

for all  $|x| \geq L$ , where  $\delta, L > 0$  and  $\alpha > 2$ . By (2) above  $A$  is not  $L^\infty$ -unique, and  $s(x) = \int_0^x s'(t) dt$  is a  $m$ -integrable and  $A$ -harmonic function (i.e., the Liouville property does not hold). Comparing this fact with P. Li's criterion in Section 3, Theorem C, we see that the drift term plays a crucial and new role for the  $L^1$ -Liouville property.

**Example 4.11** (Continuation). Let  $Af = f'' - \phi' f'$  where  $\phi$  is a locally Lipschitzian.

(4)  $\phi(x) = -c(1 + |x|)^\alpha$  where  $c > 0$  and  $\alpha > 0$ , then  $A$  is always  $L^\infty(m)$ -unique by Example 4.10(2), but it is  $L^1(m)$ -unique iff  $\alpha \leq 2$ .

(5) An example of  $\phi$  such that  $A$  is neither unique in  $L^1(m)$  nor in  $L^\infty(m)$  is furnished by  $\phi(x) = -x^3$ .

**Example 4.12** (Bessel processes). Let

$$(x_0, y_0) = (0, +\infty), \quad m'(x) = x^\gamma, \quad \gamma \in \mathbb{R},$$

and  $Af = \frac{1}{m'}(m'f')' = f'' + \frac{\gamma}{x}f'$ . We have:

- (a)  $(A, C_0^\infty(0, \infty))$  is  $L^1(m)$ -unique if and only if  $\gamma \geq 1$  [43].
- (b) Let  $p \in (1, +\infty]$ .  $(A, C_0^\infty(0, \infty))$  is  $L^p(m)$ -unique if and only if  $\gamma \leq -1$  or  $p \leq (\gamma + 1)/2$  (see [14] for  $p < +\infty$ ).

In particular  $(A, C_0^\infty(0, \infty))$  is never unique in  $L^\infty(m)$  once  $\gamma > 0$ .

Notice that when  $\gamma = d - 1$ ,  $A$  is the generator of the Bessel process  $\sqrt{2}|B_t|$  where  $B_t$  is a standard Brownian motion in  $\mathbb{R}^d$ .

## 4.2. Liouville properties

We say that  $h \in L_{\text{loc}}^1(m)$  is weakly  $A$ -harmonic (respectively subharmonic) function, if

$$\langle h, Af \rangle_m = 0 \quad (\text{respectively } \langle h, Af \rangle_m \geq 0), \quad \forall 0 \leq f \in C_0^\infty(x_0, y_0).$$

**Theorem 4.13.** Assume (4.2) and (4.3). For the operator  $A$  defined on  $C_0^\infty(x_0, y_0)$ ,

- (a) *The Liouville property for  $L^1$ -harmonic function:* “if  $h \in L^1(m)$  is weakly  $A$ -harmonic, then  $h$  is a.e. constant” holds iff either  $x_0, y_0$  are both no entrance boundary, or one of  $x_0$  or  $y_0$  is no entrance and no accessible boundary (see (4.8)).
- (b) *The Liouville property for  $L^1$ -subharmonic nonnegative function:* “if  $h \in L^1(m)$  is weakly  $A$ -harmonic and  $h \geq 0$ , then  $h$  is a.e. constant” holds iff  $x_0, y_0$  are both no entrance boundary.

**Proof.** (a) Let  $h \in L^1(m)$  be weakly  $A$ -harmonic, then we may assume without loss of generality that  $h \in C^1$  and  $h'$  is absolutely continuous and

$$ah'' + bh' = 0.$$

Hence  $h = C_1 s(x) + C_2$  for some constants  $C_1, C_2 \in \mathbb{R}$ , where  $s(x) = \int_c^x s'(t) dt$ . The Liouville property is equivalent to say that

$$\int_{x_0}^{y_0} |s(x) - C| m'(x) dx = +\infty, \quad \forall C \in \mathbb{R}. \quad (4.15)$$

*Necessity.* Assume the contrary, i.e., one of  $x_0, y_0$  is entrance but none of  $x_0, y_0$  is simultaneously no entrance and no accessible.

So we assume that  $x_0$  is entrance boundary (the same proof works if  $y_0$  is entrance). Hence  $m(x_0) > -\infty$ . This implies

$$\int_{x_0}^c |s(x) - C| m'(x) dx < +\infty, \quad \forall C \in \mathbb{R}.$$

As  $y_0$  is either entrance or accessible, we divide our discussion into two cases:

**Case 1.**  $y_0$  is entrance. In this case,  $s(x) \in L^1(m)$  and it is weakly  $A$ -harmonic, a contradiction with the Liouville property.

**Case 2.**  $y_0$  is accessible (see (4.8) for definition). In such case,  $s(y_0) < +\infty$ . Taking  $C = s(y_0)$ , we have

$$\int_c^{y_0} |s(y) - s(y_0)| m'(y) dy = \int_c^{y_0} s'(x) dx \int_c^x m'(y) dy < +\infty.$$

Thus  $s(y_0) - s(x)$  is  $m$ -integrable and weakly  $A$ -harmonic, a contradiction with the Liouville property.

*Sufficiency.* We divide its proof again into three cases.

**Case (i).**  $x_0, y_0$  are both no entrance. If for all  $C \in \mathbb{R}$ ,

$$\left| \int_{x_0}^C m'(y) dy \right| \left| \int_y^C s'(x) dx - C \right| = +\infty$$

then (4.15) holds. If the above quantity is finite for some  $C_0$ , then  $m(x_0) = -\infty$  (otherwise the above quantity is always infinite) and consequently  $C_0 = \int_{x_0}^C s'(x) dx < +\infty$ . But in this case

$$\int_c^{y_0} m'(y) dy \left| \int_y^c s'(x) dx - C_0 \right| = \int_c^x m'(y) dy \int_{x_0}^y s'(x) dx = +\infty,$$

i.e., (4.15) still holds.

**Case (ii).**  $y_0$  is no entrance and no accessible. It is enough to show that

$$\int_c^{y_0} |s(x) - C| m'(x) dx = +\infty, \quad \forall C \in \mathbb{R}.$$

Assume in contrary that the quantity above is finite for some  $C_0$ . Then  $m(y_0) < +\infty$  (otherwise the quantity above is identically infinite for  $y_0$  is no entrance) and then  $C_0$  must be  $s(y_0)$ . But

$$+\infty > \int_c^{y_0} |s(x) - C_0| m'(x) dx = \int_c^{y_0} s'(y) \int_c^y m'(y) dy$$

which is in contradiction with the fact that  $y_0$  is also no accessible.

**Case (iii).**  $x_0$  is no entrance and no accessible. The same proof as the case (ii).

(b) *Sufficiency.* (1) We first prove that if  $h$  is a weakly  $A$ -subharmonic function, then  $h$  has an absolutely continuous version  $\tilde{h}$  on  $(x_0, y_0)$  such that  $\tilde{h}'/s'$  coincides a.e. with a non-decreasing function  $p$ . Indeed,

$$\int_{x_0}^{y_0} h((1/s')f'' + (1/s')'f') dx = \langle h, Af \rangle_m \geq 0, \quad \forall 0 \leq f \in C_0^\infty(x_0, y_0).$$

Hence there is some nonnegative Radon measure  $\nu$  such that

$$\int_{x_0}^{y_0} h((1/s')f'' + (1/s')'f') dx = \int_{x_0}^{y_0} f d\nu = - \int_{x_0}^{y_0} v(x) f'(x) dx, \quad \forall f \in C_0^\infty(x_0, y_0),$$

where  $v(x)$  is some right-continuous nondecreasing function of such that  $v' = \nu$  in the sense of distribution (i.e., a primitive of  $\nu$ ). Thus  $(h/s')' = v - (1/s')' + C \in L_{\text{loc}}^1((x_0, y_0), dx)$  for some

constant  $C$ , in the sense of distribution. Consequently  $h$  has a absolutely continuous version, written again by  $h$ . Returning to the equality above, we have

$$\int_{x_0}^{y_0} v(x) f'(x) dx = - \int_{x_0}^{y_0} h(f'/s')' dx = \int_{x_0}^{y_0} (h'/s') f' dx.$$

Hence  $h'/s' = v + C$ , a.e., proving the claim.

(2) Assume in contrary that there is some non-constant nonnegative weakly  $A$ -subharmonic function  $h \in L^1(m)$ . We may assume that  $h$  is absolutely continuous by (1). Since  $h'/s' = v$  a.e. for some non-decreasing right-continuous function  $v$  by step (1),  $h$  is truly derivable on the continuous points  $x \in C_v$  of  $v$  and  $h'(x) = s'(x)v(x)$ . Hence there is a continuous point  $c_1$  of  $v$  such that  $h'(c) \neq 0$ . The proof below is completely parallel to that of Theorem 4.1.

**Case (i).**  $h'(c_1) < 0$ . In that case  $v(c_1) < 0$  and  $v(x) \leq v(c_1)$  for all  $x \in (x_0, c_1)$ . We have for all  $y \in (x_0, c_1)$ ,

$$h(y) \geq - \int_y^{c_1} s'(x)v(x) dx \geq -v(c_1) \int_y^{c_1} s'(x) dx$$

and then

$$+\infty > \int_{x_0}^{c_1} h(y)m'(y) dy \geq -v(c_1) \int_{x_0}^{c_1} m'(y) dy \int_y^{c_1} s'(x) dx,$$

a contradiction with the condition that  $x_0$  is no entrance boundary.

**Case (ii).**  $h'(c_1) > 0$ . Similarly one obtains that  $y_0$  is entrance boundary, a contradiction.

*Necessity.* By Theorem 4.1, it is enough to prove that the Liouville property in this part implies

$$h \in L^1(m), A^*h = h \implies h = 0.$$

Indeed for such  $h$ , one can prove that  $|h|$  is weakly  $A$ -subharmonic (left to the reader). Then  $|h|$  is a.e. constant. As  $h$  has an absolutely continuous version,  $h$  is constant. Hence  $h = A^*h = 0$ .  $\square$

## 5. $L^\infty$ -uniqueness of Schrödinger operators

Let us recall the definition of Kato's class.

**Definition 5.1.** Let  $V : \mathbb{R}^d \rightarrow \mathbb{R}$ . We say that  $V$  belongs to the Kato's class, denoted by  $V \in \mathcal{K}_d$  if

$$\lim_{\delta \rightarrow 0} \sup_{x \in \mathbb{R}^d} \int_{|x-y| \leq \delta} g_d(x-y) |V(y)| dy = 0,$$

where  $g_1(x) = 1$ ,  $g_2(x) = -\log|x|$  and  $g_d(x) = |x|^{2-d}$  if  $d \geq 3$ .

This class of potentials was introduced by Kato [22], and studied by Berthier, Gaveau [5]. Aizenman, Simon [1] and the survey paper of Simon [36] named this class of potential as Kato's class and presented different characterizations and applications. See the textbooks [8,9,23,33] for a rich theory.

Consider the Schrödinger operator  $A = \Delta/2 - V$  defined on  $C_0^\infty(\mathbb{R}^d)$ , where  $V$  is locally bounded and  $V^- \in \mathcal{K}_d$ . Consider the Feynman–Kac semigroup

$$P_t^V f(x) := \mathbb{E}^x f(B_t) \exp\left(-\int_0^t V(B_s) ds\right), \quad (5.1)$$

where  $((B_t), \mathbb{P}_x)$  is a Brownian motion in  $\mathbb{R}^d$  starting from  $x$ . Under the above assumption on  $V$ , it is well known that [36]:

$$\sup_{t \leq T} \|P_t^V\|_\infty < +\infty, \quad \lim_{t \rightarrow 0^+} \|P_t^V\|_\infty = 1.$$

Since  $(P_t^V)$  is strongly continuous on  $L^1(\mathbb{R}^d, dx)$ , it is also strongly continuous on  $L^p$  for every  $p \in [1, +\infty]$  (w.r.t.  $\mathcal{C}(L^\infty, L^1)$  if  $p = +\infty$ ). Letting  $A_t = \int_0^t V(B_s) ds$ , by Ito's formula, we have for  $f \in C_0^\infty(\mathbb{R}^d)$ ,

$$d(f(B_t)e^{-A_t}) = e^{-A_t} \nabla f(B_t) \cdot dB_t + \frac{1}{2} \Delta f(B_t) dt - V(B_t)e^{-A_t} f(B_t) dt.$$

Since the local martingale  $M_T = \int_0^T e^{-A_t} \nabla f(B_t) \cdot dB_t$  satisfies

$$\mathbb{E}^x[M]_T = \mathbb{E}^x \int_0^T e^{-2A_t} |\nabla f|^2(B_t) \cdot dt \leq T \sup_{t \leq T} \|P_t^{2V}\|_\infty \|\nabla f\|_\infty^2, \quad \forall T > 0,$$

it is a true martingale. Thus taking expectation under  $\mathbb{P}_x$  in the above formula we get

$$P_T^V f(x) - f(x) = \int_0^T P_t^V (\Delta - V) f(x) dt, \quad \forall T \geq 0,$$

which means that the generator  $\mathcal{L}_{(\infty)}^V$  of  $(P_t^V)$  in  $L^\infty$  extends  $A$ .

The objective of this section is to prove

**Theorem 5.1.** *Let  $V \in L_{\text{loc}}^\infty(\mathbb{R}^d, dx)$  such that  $V^- \in \mathcal{K}_d$ . Then the Schrödinger operator  $(-\Delta/2 + V, C_0^\infty(\mathbb{R}^d))$  is  $L^\infty(\mathbb{R}^d, dx)$ -unique. In particular the generator  $\mathcal{L}_{(1)}^V$  of  $(P_t^V)$  in  $L^1(\mathbb{R}^d, dx)$  can be identified as*

$$\mathbb{D}(\mathcal{L}_{(1)}^V) = \left\{ g \in L^1(\mathbb{R}, dx); \frac{1}{2} \Delta g - Vg \in L^1(\mathbb{R}, dx) \right\},$$

$$\mathcal{L}_{(1)}^V g = \frac{1}{2} \Delta g - Vg, \quad \forall g \in \mathbb{D}(\mathcal{L}_{(1)}^V). \quad (5.2)$$

When  $L^\infty$  is replaced by  $L^p$  with  $p \in [1, +\infty)$ , the statement above is the well-known Kato's theorem for  $p = 2$ , see [9, Corollary 2.2]; for  $p \in (1, +\infty)$ , it should be known to specialists longtime ago (see [45] for extension to the infinite-dimensional setting); and for  $p = 1$ , it is contained in [42].

**Proof of Theorem 5.1.** The existence assumption in Theorem 2.1 is satisfied by  $A$ , as shown previously. By Theorem 2.1, it is enough to show that for any

$$\lambda < \lambda(V) = \inf \left\{ \frac{1}{2} \int |\nabla f|^2 + V f^2 dx; f \in C_0^\infty(\mathbb{R}^d), \|f^2\|_2 = 1 \right\}$$

(the lowest energy of the Schrödinger operator), if  $h \in L^q(\mathbb{R}^d, dx)$  verifies  $\frac{1}{2} \Delta h - Vh + \lambda h = 0$  in distribution, then  $h = 0$ .

As  $V$  is locally bounded, by [1, Theorem 1.5],  $h$  admits a continuous version and then it may and will be assumed to be continuous. By the mean value theorem due to Aizenman, Simon [1, Corollary 3.9]: there is some constant  $C > 0$  such that

$$|h(x)| \leq C \int_{|y| \leq 1} |h(x-y)| dy, \quad \forall x \in \mathbb{R}^d.$$

As  $h \in L^1(\mathbb{R}^d)$ ,  $h$  is globally bounded and then  $h \in L^2(\mathbb{R}^d, dx)$ . Now by the  $L^2$ -uniqueness of  $\frac{1}{2} \Delta - V$  (recalled above) and Theorem 2.1,  $h$  belongs to the domain  $\mathbb{D}(\mathcal{L}_{(2)}^V)$  of the generator  $\mathcal{L}_{(2)}^V$  of  $(P_t^V)$  on  $L^2$  and  $\mathcal{L}_{(2)}^V h = \frac{1}{2} \Delta h - Vh = -\lambda h$ . Hence  $P_t^V h = e^{-\lambda t} h$  for all  $t > 0$ , which is possible only for  $h = 0$ , because  $\lambda < \lambda(V)$  and  $\|P_t^V\|_2 = e^{-\lambda(V)t}$ .  $\square$

## 6. Multi-dimensional diffusions: comparison with one-dimensional diffusion

In this section we consider the operator

$$A = \frac{1}{2} \Delta f + b \nabla f, \quad \forall f \in \mathbb{D}(A) = C_0^\infty(\mathbb{R}^d), \quad (6.1)$$

where  $d \geq 2$  and the vector field  $b: \mathbb{R}^d \rightarrow \mathbb{R}^d$  is measurable and locally bounded. By the Girsanov formula, for every  $x \in \mathbb{R}^d$ , there is a unique martingale solution  $((X_t)_{0 \leq t < e})$ , where  $e$  is the explosion time, i.e., on some probability space equipped with a Brownian motion  $(B_t)$ ,  $(X_t)$  verifies

$$X_0 = x, \quad dX_t = dB_t + b(X_t) dt, \quad \forall t < e = \sup_{R>0} e_R,$$

where  $e_R = \inf\{t; |X_t| > R\}$  and the law of  $(X_t)_{0 \leq t < e}$  is unique. Let  $P_t f(x) = \mathbb{E}^x f(X_t) 1_{t < e}$ . The Girsanov formula gives us

$$P_t g(x) = \mathbb{E} \exp \left( \int_0^t b(x + B_t) dB_t - \frac{1}{2} \int_0^t |b|^2(x + B_t) dt \right) g(x + B_t) \quad (6.2)$$

for every measurable function  $g \geq 0$ . Consequently  $P_t(x, dy) = p_t(x, y) dy$  and  $p_t(x, y) > 0$ ,  $dy$ -a.e.

### 6.1. $L^\infty$ -uniqueness

If there is some measurable locally bounded function  $\beta: \mathbb{R}^+ \rightarrow \mathbb{R}$  such that

$$\frac{b(x) \cdot x}{|x|} \geq \beta(|x|), \quad \forall 0 \neq x \in \mathbb{R}^d, \quad (6.3)$$

then for every initial point  $x \neq 0$ ,

$$|X_t| - |x| \geq \int_0^t \left( \beta(|X_t|) + \frac{d-1}{2|X_t|} \right) dt + \text{a real Brownian motion}, \quad \forall t < e$$

(the diffusion never reaches the origin  $o$  for  $d \geq 2$ ). In other words,  $|X_t|$  go to infinity more rapidly than the one-dimensional diffusion generated by

$$A_1 = \frac{1}{2} \frac{d^2}{dr^2} + \tilde{\beta}(r) \frac{d}{dr}, \quad \tilde{\beta}(r) = \beta(r) + \frac{d-1}{2r}.$$

This is standard in probability, see Malliavin [30] and Ikeda, Watanabe [20]. Hence if  $+\infty$  is a no entrance boundary for  $A_1$ , i.e.,

$$\int_0^\infty m'(y) dy \int_1^y s'(r) dr = +\infty, \quad (6.4)$$

where

$$s'(r) = \exp \left( - \int_1^r 2\tilde{\beta}(t) dt \right), \quad m'(x) = \frac{1}{s'(x)} = r^{d-1} \exp \left( \int_1^r 2\beta(t) dt \right), \quad (6.5)$$

are respectively the derivatives of the scale and speed function of  $A_1$ , one may guess naturally that the point  $\partial$  at infinity of  $\mathbb{R}^d$  is in some sense a no entrance boundary for  $A$ , and  $A$  is  $L^\infty(dx)$ -unique in the spirit of Theorem 4.1. That good sense is correct, as confirmed by

**Theorem 6.1.** Assume (6.3) and (6.4), then the operator  $A$  given by (6.1) is  $L^\infty(\mathbb{R}^d, dx)$ -unique. In particular for every  $f(x) \in L^1(\mathbb{R}^d, dx)$ , the Fokker–Planck equation

$$\partial_t u(t, x) = \frac{1}{2} \Delta u(t, x) - \operatorname{div}(ub), \quad u(0, x) = f(x) \in L^1(\mathbb{R}^d, dx)$$

has a unique  $L^1(dx)$ -weak solution (see Theorem 0.2 for its definition).

This theorem is sharp in view of the necessary and sufficient condition in Theorem 4.1 in the one-dimensional case.

**Proof.** Step 1. At first  $(P_t)$  is a  $C_0$ -semigroup on  $L^\infty(dx)$  w.r.t. the topology  $\mathcal{C}(L^\infty, L^1)$  by Proposition 3.3 and the fact that  $P_t(x, dy)$  is absolutely continuous w.r.t.  $dy$ .

Putting  $X_t = \partial$  after the explosion time ( $t \geq e$ ) and  $g(\partial) = 0$  for  $g: \mathbb{R}^d \rightarrow \mathbb{R}$ , by Ito's formula we have for every  $f \in C_0^\infty(\mathbb{R}^d)$ ,

$$f(X_t) - f(x) - \int_0^t Af(X_s) ds$$

is a local martingale. As it is bounded over bounded time intervals, it is a true martingale. Thus by taking expectation under  $\mathbb{P}_x$ ,

$$P_t f(x) - f(x) = \int_0^t P_s f(x) ds, \quad \forall t \geq 0.$$

Then  $A \subset \mathcal{L}(\infty)$ , the generator of  $(P_t)$  in  $L^\infty(dx)$ .

Step 2. By Theorem 2.1, for the  $L^\infty$ -uniqueness of  $A$  it is enough to show that if  $u \in L^1(dx)$  verifies

$$\langle u, (A - 1)f \rangle = 0, \quad \forall f \in C_0^\infty(\mathbb{R}^d),$$

then  $u = 0$ , here and hereafter  $\langle f, g \rangle := \int_{\mathbb{R}^d} fg dx$ . By the ellipticity regularity result in [14, Chapter II, Lemma 2.2],  $u \in L_{\text{loc}}^\infty$  and  $\nabla u \in L_{\text{loc}}^d \subset L_{\text{loc}}^2$ . By the fact that  $C_0^\infty(\mathbb{R}^d)$  is dense in

$$\{f \in L^2; \nabla f \in L^2, \text{ and the support of } f \text{ is compact}\},$$

an integration by parts yields

$$-\frac{1}{2} \langle \nabla u, \nabla f \rangle + \langle u, b \cdot \nabla f \rangle - \langle u, f \rangle = 0$$

for all  $f \in H^{1,2}(\mathbb{R}^d)$  with compact support. Now one can follow Eberle [14, Chapter II, proof of Theorem 2.5, Step 2] to show that (an inequality of Kato's type)

$$-\frac{1}{2} \langle \nabla |u|, \nabla f \rangle + \langle |u|, b \cdot \nabla f \rangle - \langle |u|, f \rangle \geq 0 \quad (6.6)$$



for all  $f \in H^{1,2}(\mathbb{R}^d)$  with compact support.

Step 3. Let  $G(r) = \int_{B(r)} |u| dx$ , where  $B(r) = \{x \in \mathbb{R}^d, |x| \leq r\}$ .  $G$  is absolutely continuous and

$$G'(r) = \int_{\partial B(r)} |u| d_\sigma x, \quad dr\text{-a.e.},$$

where  $d_\sigma x$  is the surface measure on the sphere  $\partial B(r)$  (the boundary of  $B(r)$ ).

Now for every  $0 < r_1 < r_2$ , taking  $f = \min\{r_2 - r_1, (r_2 - |x|)^+\}$  in (6.6) and setting  $\gamma(x) = x/|x| = \nabla|x|$ , we get

$$\begin{aligned} & \int_{B(r_2) \setminus B(r_1)} \left( \frac{1}{2} \nabla |u| \cdot \gamma(x) - |u| b(x) \cdot \gamma(x) \right) dx \\ & \geq \int_{B(r_2) \setminus B(r_1)} |u| (r_2 - |x|) dx + G(r_1)(r_2 - r_1). \end{aligned} \quad (6.7)$$

Since

$$\nabla |u| \cdot \gamma = \operatorname{div}(|u| \gamma) - |u| \operatorname{div}(\gamma) = \operatorname{div}(|u| \gamma) - |u| \frac{d-1}{|x|}$$

then by the Gauss–Green formula we have for  $dr_1 \otimes dr_2$ -a.e.,  $0 < r_1 < r_2$ ,

$$\int_{B(r_2) \setminus B(r_1)} \nabla |u| \cdot e(x) dx = G'(r_2) - G'(r_1) - (d-1) \int_{r_1}^{r_2} \frac{1}{r} G'(r) dr.$$

Using our condition (6.3) and Fubini's theorem we have

$$\begin{aligned} & \int_{B(r_2) \setminus B(r_1)} -|u| b(x) \cdot \gamma(x) dx \leq - \int_{r_1}^{r_2} G'(r) \beta(r) dr \quad \text{and} \\ & \int_{B(r_2) \setminus B(r_1)} |u| (r_2 - |x|) dx = \int_{r_1}^{r_2} (r_2 - r) G'(r) dr = \int_{r_1}^{r_2} \int_{r_1}^r G'(t) dt dr. \end{aligned}$$

Substituting those into (6.7) we obtain

$$\frac{1}{2} (G'(r_2) - G'(r_1)) - \int_{r_1}^{r_2} \tilde{\beta}(r) G'(r) dr \geq \int_{r_1}^{r_2} G(r) dr$$

for  $dr_1 \otimes dr_2$ -a.e.  $0 < r_1 < r_2$ . That means

$$A_1^- G := \frac{1}{2} G''(r) - \tilde{\beta}(r) G'(r) \geq G(r) \quad (6.8)$$

in distribution on  $(0, +\infty)$ .

*Step 4.* Notice that the sign of  $\tilde{\beta}$  in  $A_1^-$  is negative, opposite to the operator  $A_1$  and then the derivative of the scale (respectively speed) function of  $A_1^-$  is exactly  $m'(x)/2$  (respectively  $2s'(x)$ ), where  $m'(x)$  (respectively  $s'(x)$ ) is the derivative of the speed (respectively scale) function of  $A_1$ , introduced in (6.4). Hence we can write (6.8) in the Feller's form,

$$(G'/m')' \geq s'G.$$

Assume now in contrary that  $u \neq 0$ , i.e.,  $G(r_0) > 0$  for some  $r_0 > 0$ . The above inequality in distribution implies that for  $dr$ -a.e.  $r > r_0$ ,

$$G'(r) \geq m'(r) \int_{r_0}^r s'(t) G(t) dt \geq m'(r) G(r_0) \int_{r_0}^r s'(t) dt,$$

where it follows,

$$\int_{\mathbb{R}^d} |u| dx = G(+\infty) \geq G(r_0) \int_{r_0}^{+\infty} m'(r) dr \int_{r_0}^r s'(t) dt$$

which is infinite by our assumption that  $+\infty$  is a no entrance boundary. This is in contradiction with the assumption that  $u \in L^1(\mathbb{R}^d, dx)$ .  $\square$

The following corollary was announced in [46].

**Corollary 6.2.** Assume that for some nondecreasing function  $\xi: \mathbb{R} \rightarrow [1, +\infty)$  such that

$$b(x) \cdot x/|x| \geq -\xi(|x|), \quad dx\text{-a.e.}, \quad \text{and} \quad \int_0^{+\infty} \frac{1}{\xi(r)} dr = +\infty$$

then  $A$  is  $L^\infty(\mathbb{R}^d, dx)$ -unique.

**Proof.** This is a direct consequence of Theorem 6.1 and Corollary 4.9.  $\square$

**Remark 6.3.** For  $A = \frac{1}{2}(\Delta - \nabla\phi \cdot \nabla)$  which is symmetric on  $L^2(\mu = e^{-\phi} dx)$  (the so called Nelson's diffusion operator or generalized Schrödinger operator), the results on its  $L^p$ -uniqueness are numerous, see Wielens [41], Liskevitch [28] and Liskevitch, Semenov [29], etc. for the  $L^2$ -uniqueness, Stannat [37] and the first named author [42–44] for the  $L^1$ -uniqueness.

The  $L^p$ -uniqueness of  $A$  for  $1 \leq p < +\infty$  in the general setting was studied by Eberle [14] extensively.

## 6.2. Liouville properties

With exactly the same proof we have the following Liouville property:

**Proposition 6.4.** Assume (6.3) and (6.4).

- (a) Let  $\lambda > 0$ . If  $u \in L^1(dx)$  is nonnegative and weakly  $(A^* - \lambda)$ -subharmonic with locally finite energy, i.e.,

$$\langle u, (A - \lambda)f \rangle \geq 0, \quad \forall 0 \leq f \in C_0^\infty(\mathbb{R}^d),$$

and  $\nabla u \in L_{\text{loc}}^2(\mathbb{R}^d, dx)$ , then  $u = 0$ .

- (b) If

$$m(+\infty) = \int_1^\infty r^{d-1} \exp\left(\int_1^r 2\beta(t) dt\right) dr = +\infty \quad (6.9)$$

(which is stronger than (6.4)) then every nonnegative  $A^*$ -subharmonic function  $u \in L^1(dx)$  with locally finite energy is zero.

The condition (6.9) means that the invariant measure  $m$  for the one-dimensional operator  $A_1$  is infinite. If  $m$  is finite,  $A$  may possess positive  $A^*$ -harmonic function  $h \in L^1(dx)$ , i.e.,  $h dx$  is a  $A$ -invariant measure (a very current situation in probability).

**Definition 6.1.** A Radon measure  $\nu$  is called  $A$ -invariant (sub-invariant), if

$$\langle \nu, Af \rangle = 0 \quad (\text{respectively } \geq 0), \quad \forall 0 \leq f \in C_0^\infty(\mathbb{R}^d).$$

**Remark 6.5.** For a  $A$ -invariant measure  $\nu$  with finite variation (i.e.,  $\nu \in M_b(\mathbb{R}^d)$ ), it is not necessary that  $\nu P_t = \nu$  for all  $t \geq 0$ . This question is essentially related with the  $L^\infty$ -uniqueness of  $A$ .

For the existence, uniqueness and the regularity of the invariant measure, the reader is referred to [20] for  $b$  smooth, and [6] and the references therein for  $b$  singular.

**Theorem 6.6.** Assume (6.3) and (6.4). Then the linear space  $\mathcal{I}_A$  of  $A$ -invariant measures  $\nu$  with finite variation (i.e.,  $\nu \in M_b(\mathbb{R}^d)$ ) is of at most dimension one. If its dimension is one, then there is some  $A$ -invariant probability measure  $\mu$  (so any  $A$ -invariant measure  $\nu \in M_b(\mathbb{R}^d)$  is of form  $c\mu$  with  $c \in \mathbb{R}$ ),  $\mu P_t = \mu$  for all  $t \geq 0$ , and furthermore  $(P_t)$  is conservative, i.e.,  $P_t 1 = 1$  on  $\mathbb{R}^d$ .

**Proof.** Assume that  $\dim(\mathcal{I}_A) \geq 1$ . Let  $\nu$  be a nonzero  $A$ -invariant measure with finite variation. Then  $\nu = h dx$  (by [14, Chapter II, Lemma 2.2]) and  $\langle h, Af \rangle = 0$  for all  $f \in C_0^\infty(\mathbb{R}^d)$ , i.e.,  $A^*h = 0$  ( $A^*$  be the dual operator of  $A$  w.r.t.  $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_{dx}$ ). By the  $L^\infty$ -uniqueness in Theorems 6.1 and 2.1,  $h \in \mathbb{D}(\mathcal{L}_{(1)}^*)$ , the domain of the generator of the dual semigroup  $P_t^*$  on  $L^1(dx)$ , and  $\mathcal{L}_{(1)}^* h = 0$ . Hence we have  $P_t^* h = h$  or equivalently  $\nu P_t = \nu$  for all  $t \geq 0$ . Let  $|\nu| := |h| dx$ . By the sub-Markov property of  $P_t$ , we have

$$|\nu| \leq |\nu| P_t, \quad \text{but} \quad (|\nu| P_t)(\mathbb{R}^d) \leq |\nu|(\mathbb{R}^d).$$

Thus  $|v| = |v|P_t$  for all  $t \geq 0$ , i.e.,  $\mu := |v|/v(\mathbb{R}^d)$  is an invariant probability measure for  $(P_t)$  (it is then  $A$ -invariant). Consequently  $v^\pm = (|v| \pm v)/2$  are also  $(P_t)$ -invariant.

Since for every  $x$ ,  $P_t(x, dy) = p_t(x, y) dy$  with  $p_t(x, y) > 0$ ,  $dy$ -a.e., the invariant probability measure  $\mu$  of  $(P_t)$  is unique, and  $\mu \sim dx$ . Thus  $v^\pm = c^\pm \mu$  for some constants  $c^+$ ,  $c^-$ . Hence  $v = (c^+ - c^-)\mu$ . That proves  $\dim(\mathcal{I}_A) = 1$ .

Furthermore, for any  $x \in \mathbb{R}^d$ ,

$$P_t 1(x) = P_{t/2}(P_{t/2} 1)(x) = 1,$$

the desired conservativeness.  $\square$

The idea of comparison of a multi-dimensional diffusion with a one-dimensional diffusion is very old and fruitful for various properties, see Malliavin [30] for asymptotic behavior of the Green's function, Ikeda and Watanabe [20] for non-explosion, recurrence, etc. It seems to be the first time that this idea of comparison was carried out for the uniqueness of  $A$ -invariant measure and the  $L^1$ -uniqueness of the Fokker–Planck equation.

The criterion above on the uniqueness of  $A$ -invariant measure (a version of Liouville theorem) seems to be better than all known results of the same type (i.e., basing on the growth of  $b(x) \cdot x$ ) mentioned in [6,20], and it is sharp by Theorem 4.13. The idea in this section will be developed for diffusion operators over a Riemannian manifold in a forthcoming work.

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